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Mathematics

On *XI*-Semilattices of Unions

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ABSTRACT. In this article some *XI*-semilattices of unions are described. They are found when studying complete semigroups of binary relations defined by complete *X*-semilattices of unions. Knowing all *XI*-subsemilattices of the given complete *X*-semilattice of unions, we can characterize all the idempotents of the given semigroup.

Key words: *semilattice, semigroup of binary relations.*

Definition 1. Let D be a complete X-semilattice of unions, $\varnothing \neq D' \subseteq D$ and $N(D,D') = \{Z \in D \mid Z \subseteq Z' \text{ for any } Z' \in D'\}$. It is evident that $N(D,D')$ is a set of all lower bounds of nonempty sets D' in semilattice D. If $N(D,D') \neq \varnothing$, then $\cup N(D,D') \in D$ is the greatest lower bound of the set D' in D. Denote this element by symbol $\Lambda(D,D')$, i.e. $\Lambda(D,D') = \cup N(D,D')$.

Note, if the element $\Lambda(D,D')$ in semilattice D exists, then we write $\Lambda(D,D') \in D$, and vice versa – $\Lambda(D,D') \notin D$.

Definition 2. Let $t \in \breve{D} = \cup D$ and $D_t = \{Z \in D \mid t \in Z\}$. It is said that complete X-semilattice of unions D is XI-semilattice of unions, if it satisfies the following conditions:

a) $\Lambda(D,D_t) \in D$, for any $t \in \breve{D}$;

b) $Z = \bigcup_{t \in Z} \Lambda(D,D_t)$, for any nonempty element Z of the semilattice D.

True statement 1) and 2) can be found in [2 or 3].

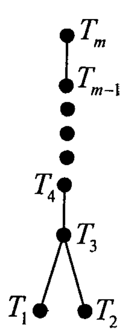

Fig. 1.

1) Let $D = \{\breve{D}, Z_1, Z_2, \dots, Z_{m-1}\}$ be some finite X-semilattice of unions and $C(D) = \{C, P_1, P_2, \dots, P_{m-1}\}$ is the family of sets of pairwise nonintersected subsets of the set X. If χ is mapping of the semilattice D on the family of sets $C(D)$, satisfying the condition $\chi(\breve{D}) = C$ and $\chi(Z_i) = P_i$, for any i=1,2,...,m–1 and $D_Z = D \setminus \{T \in D \mid Z \subseteq T\}$, then the following equalities are true:

$$\breve{D} = C \cup P_1 \cup P_2 \cup \dots \cup P_{m-1}, \quad Z_i = C \cup \bigcup_{T \in D_{Z_i}} \chi(T). \tag{1}$$

Note that the equalities of type 1 are called formal.

2) When presenting elements of semilattice D in type (1) among the parameters P_i (i=1,2,...,m–1) there exist such that for the given semilattice D can not be empty sets. Such sets P_i ($1 \leq i \leq m$–1) are

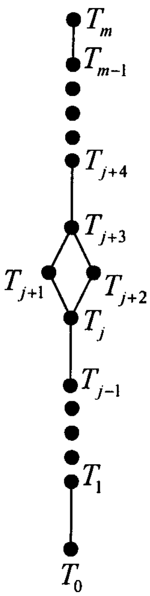

Fig. 2.

called basis sources, and those sets P_j ($1 \le j \le m-1$) that can be also empty sets will be called the sources of completeness. It is proved that the number of covering elements of the prototype of basis source at mapping χ is always equal to one, and a number of covering elements of the prototype of the completeness source at mapping χ either does not exist or is always bigger than one.

Note, that set C is always considered to be the source of completeness.

Theorem 1. *Let* $Q = \{T_1, T_2, T_3, ..., T_m\}$ $(m \ge 3)$ *be such subsemilattice of semilattice D, that* $T_1, T_2 \notin \{\varnothing\}$, $T_1 \cup T_2 = T_3$ *and* $T_3 \subset T_4 \subset ... \subset T_m$ *(see Fig. 1). Then Q is XI-semilattice of unions if and only if* $T_1 \cap T_2 = \varnothing$.

Proof. Let $P_1, P_2, ..., P_{m-1}$ and C be pairwise nonintersected subsets of the set X and φ be mapping of semilattice Q on the set $\{P_1, P_2, ..., P_{m-1}, C\}$, having the form:

$$\varphi = \begin{pmatrix} T_1 & T_2 & ... T_{m-1} & T_m \\ P_1 & P_2 & ... P_{m-1} & C \end{pmatrix}.$$

Then the formal equality corresponding to the semilattice Q is as follows:

$$\begin{array}{l} T_m = C \cup P_1 \cup P_2 \cup ... \cup P_{m-1}, \quad T_{m-1} = C \cup P_1 \cup P_2 \cup ... \cup P_{m-2}, \\ ---------------------------------------` \\ T_3 = C \cup P_1 \cup P_2, \quad T_2 = C \cup P_1, \; T_1 = C \cup P_2, \end{array} \tag{2}$$

where $|C| \ge 0$ and $|P_i| \ge 1$ for any $i = 1,2,..., m-1$. Further, let $t \in T_m$. Then from the equalities (2) we get:

$$Q_t = \begin{cases} Q, \; if \;\; t \in C, \\ \{T_2, T_3, ..., T_m\}, \; if \;\; t \in P_1, \\ \{T_1, T_3, ..., T_m\}, \; if \;\; t \in P_2, \\ ------------- \\ \{T_{m-1}, T_m\}, \; if \;\; t \in P_{m-2}, \\ \{T_m\}, \; if \;\; t \in P_{m-1}, \end{cases} \quad \text{and} \quad \Lambda(Q, Q_t) = \begin{cases} T_2, \; if \;\; t \in P_1, \\ T_1, \; if \;\; t \in P_2, \\ -------- \\ T_{m-1}, \; if \;\; t \in P_{m-2}, \\ T_m, \; if \;\; t \in P_{m-1}, \end{cases}$$

We received $\Lambda(Q, Q_t) \notin Q$, if $t \in C$. Proceeding from Definition 2, at $C \ne \varnothing$, it follows that semilattice Q can not be *XI*-semilattice.

Then we suppose that $|C| \ge 0$. Thus we can assume that $C = \varnothing$. In this case we shall have $\Lambda(Q, Q_t) \in Q$, for $\Lambda(Q, Q_t) \in Q$, any $t \in T_m$. Besides it is easily checked that any nonempty element of semilattice Q is the union of some elements of $\Lambda(Q, Q_t)$, where $t \in T_m$. Now taking into account Definition 2, we get that Q is *XI*-semilattice of unions.

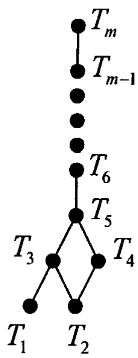

Fig. 3.

Note that proceeding from the equality (2) condition $C = \varnothing$ is fulfilled if and only if $T_1 \cap T_2 = \varnothing$.

The necessity of the conditions of the given theorem is proved. Let us prove the sufficiency of the conditions of the given theorem.

In fact if $T_1 \cap T_2 = \varnothing$, then out of formal equalities (2) it follows that $C = \varnothing$. Proceeding from the last condition we have $\Lambda(Q, Q_t) \in Q$ for any $t \in T_m$ and any nonempty element of semilattice Q is a union of some elements of the form $\Lambda(Q, Q_t)$, where $t \in T_m$.

The Theorem is proved.

Theorem 2. *Let* $Q = \{T_0, T_1, T_2, T_3, ..., T_m\}$ $(m \ge 3)$ *be such subsemilattice of semilattice D and j be such fixed natural number that* $0 \le j \le m-3$ *and*

$$T_0 \subset T_1 \subset ... \subset T_j \subset T_{j+1} \subset T_{j+3} \subset ... \subset T_m, \quad T_0 \subset T_1 \subset ... \subset T_j \subset T_{j+2} \subset T_{j+3} \subset ... \subset T_m,$$

$T_{j+1} \setminus T_{j+2} \neq \varnothing, \quad T_{j+2} \setminus T_{j+1} \neq \varnothing, \quad T_{j+1} \cup T_{j+2} = T_{j+3}$

(see Fig. 2). *Then Q is always XI-semilattice of unions.*

Proof. Let $P_0, P_1, ..., P_{m-1}$ and C be pairwise nonintersected subsets of the set X and φ be mapping of semilattice Q on the set $\{P_0, P_1, ..., P_{m-1}, C\}$, having the form:

$$\varphi = \begin{pmatrix} T_0 \; T_1 \; T_2 \; ... T_{j-1} \; T_j \; T_{j+1} \; T_{j+2} \; T_{j+3} \; ... T_{m-1} \; T_m \\ P_0 \; P_1 \; P_2 \; ... P_{j-1} \; P_j \; P_{j+1} \; P_{j+2} \; P_{j+3} \; ... P_{m-1} \; C \end{pmatrix}.$$

Then formal equalities corresponding to the semilattice Q has the form:

$$\begin{aligned}
&T_m = C \cup P_0 \cup P_1 \cup P_2 \cup ... \cup P_{j-1} \cup P_j \cup P_{j+1} \cup P_{j+2} \cup P_{j+3} \cup ... \cup P_{m-1}, \\
&T_{m-1} = C \cup P_0 \cup P_1 \cup P_2 \cup ... \cup P_{j-1} \cup P_j \cup P_{j+1} \cup P_{j+2} \cup P_{j+3} \cup ... \cup P_{m-2}, \\
&------------------------------ \\
&T_{j+3} = C \cup P_0 \cup P_1 \cup P_2 \cup ... \cup P_{j-1} \cup P_j \cup P_{j+1} \cup P_{j+2}, \\
&T_{j+2} = C \cup P_0 \cup P_1 \cup P_2 \cup ... \cup P_{j-1} \cup P_j \cup P_{j+1}, \\
&T_{j+1} = C \cup P_0 \cup P_1 \cup P_2 \cup ... \cup P_{j-1} \cup P_j \cup P_{j+2}, \\
&T_j = C \cup P_0 \cup P_1 \cup P_2 \cup ... \cup P_{j-1}, \\
&------------------------------ \\
&T_2 = C \cup P_0 \cup P_1, \quad T_1 = C \cup P_0, \; T_0 = C,
\end{aligned} \tag{3}$$

where $|C| \geq 0$, $|P_j| \geq 0$ and $|P_i| \geq 1$ for any i = 0,1,2,...,j–1, j+1,..., m–1. Further, let $t \in T_m$. Then from the equalities (3) we get:

$$Q_t = \begin{cases} Q, \; if \;\; t \in C, \\ \{T_1, T_2, ..., T_m\}, \; if \;\; t \in P_0, \\ \{T_2, T_3, ..., T_m\}, \; if \;\; t \in P_1, \\ ------------ \\ \{T_j, T_{j+1}, ..., T_m\}, \; if \;\; t \in P_{j-1}, \\ \{T_{j+1}, T_{j+2}, ..., T_m\}, \; if \;\; t \in P_j, \\ \{T_{j+2}, T_{j+3}, ..., T_m\}, \; if \;\; t \in P_{j+1}, \\ \{T_{j+1}, T_{j+3}, ..., T_m\}, \; if \;\; t \in P_{j+2}, \\ ------------ \\ \{T_{m-1}, T_m\}, \; if \;\; t \in P_{m-2}, \\ \{T_m\}, \; if \;\; t \in P_{m-1}, \end{cases} \quad \text{and} \quad \Lambda(Q, Q_t) = \begin{cases} T_0, \; if \;\; t \in C, \\ T_1, \; if \;\; t \in P_0, \\ T_2, \; if \;\; t \in P_1, \\ -------- \\ T_j, \; if \;\; t \in P_{j-1}, \\ T_j, \; if \;\; t \in P_j, \\ T_{j+2}, \; if \;\; t \in P_{j+1}, \\ T_{j+1}, \; if \;\; t \in P_{j+2} \\ -------- \\ T_{m-1}, \; if \;\; t \in P_{m-2}, \\ T_m, \; if \;\; t \in P_{m-1}, \end{cases}$$

We get that $\Lambda(Q, Q_t) \in Q$, for any $t \in T_m$.

It is easily checked that any nonempty element of semilattice Q is a union of some elements of the type $\Lambda(Q, Q_t)$, where $t \in T_m$. Now taking into account Definition 2, we get that Q is XI-semilattice of unions.

The Theorem is proved.

Theorem 3. *Let* $Q = \{T_1, T_2, T_3, ..., T_m\}$ $(m \geq 5)$ *be such subsemilattice of semilattice D, that*

$T_1 \subset T_3 \subset T_5 \subset \ldots \subset T_m$, $T_2 \subset T_3 \subset T_5 \subset \ldots \subset T_m$, $T_2 \subset T_4 \subset T_5 \subset \ldots \subset T_m$,

$T_1 \setminus T_2 \neq \varnothing$, $T_2 \setminus T_1 \neq \varnothing$, $T_3 \setminus T_4 \neq \varnothing$, $T_4 \setminus T_3 \neq \varnothing$, $T_1 \cup T_2 = T_3$, $T_3 \cup T_4 = T_5$

(see Fig. 3). *Then Q is XI-semilattice of unions in that case if and only if* $T_1 \cap T_4 = \varnothing$.

Proof. Let $P_1, P_2, \ldots, P_{m-1}$ and C be pairwise nonintersected subsets of the set X and *j* be mapping of semilattice Q on the set $\{P_1, P_2, \ldots, P_{m-1}, C\}$, having the form:

$$\varphi = \begin{pmatrix} T_1 \; T_2 \ldots T_{m-1} \; T_m \\ P_1 \; P_2 \ldots P_{m-1} \;\; C \end{pmatrix}.$$

Then formal equalities corresponding to the semilattice Q have the form:

$$\begin{aligned} &T_m = C \cup P_1 \cup P_2 \cup \ldots \cup P_{m-1}, \quad T_{m-1} = C \cup P_1 \cup P_2 \cup \ldots \cup P_{m-2}, \\ &------------------------------------- \\ &T_5 = C \cup P_1 \cup P_2 \cup P_3 \cup P_4, \quad T_4 = C \cup P_1 \cup P_2 \cup P_3, \\ &T_3 = C \cup P_1 \cup P_2 \cup P_4, \quad T_2 = C \cup P_1, \quad T_1 = C \cup P_2 \cup P_4, \end{aligned} \tag{4}$$

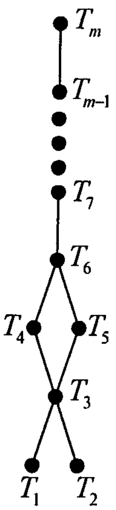

Fig. 4.

where $|C| \geq 0$, $|P_2| \geq 0$ and $|P_i| \geq 1$ for any $i = 1,3,\ldots,j-1, j+1, \ldots m$-1. Then from the equalities (4) we receive:

$$Q_t = \begin{cases} Q, \; if \;\; t \in C, \\ \{T_2, T_3, \ldots, T_m\}, \; if \;\; t \in P_1, \\ \{T_1, T_3, \ldots, T_m\}, \; if \;\; t \in P_2, \\ ------------- \\ \{T_{m-1}, T_m\}, \; if \;\; t \in P_{m-2}, \\ \{T_m\}, \; if \;\; t \in P_{m-1}, \end{cases} \quad \text{and} \quad \Lambda(Q, Q_t) = \begin{cases} T_2, \; if \;\; t \in P_1, \\ T_1, \; if \;\; t \in P_2, \\ -------- \\ T_{m-1}, \; if \;\; t \in P_{m-2}, \\ T_m, \; if \;\; t \in P_{m-1}, \end{cases}$$

We get that $\Lambda(Q, Q_t) \notin Q$, if $t \in C \cup P_2$. Proceeding from Definition 2 at $C \cup P_2 \neq \varnothing$ it follows that semilattice Q can not be *XI*-semilattice.

Further, we assume that $|C| \geq 0$ and $|P_2| \geq 0$. Thus we may consider that $C = P_2 = \varnothing$. In this case we have $\Lambda(Q, Q_t) \in Q$ for any $t \in T_m$. Besides, it is easily checked that any nonempty element of semilattice Q is a union for some elements of the type $\Lambda(Q, Q_t)$, where $t \in T_m$. Now taking into account Definition 2, we get that Q is *XI*-semilattice of unions.

Note that on the basis of equality (4) condition $C = P_2 = \varnothing$ is fulfilled if and only if $T_1 \cap T_4 = \varnothing$.

The necessity of the conditions of the given theorem is proved. Let us prove the sufficiency of the conditions of the given theorem. In fact, if $T_1 \cap T_4 = \varnothing$, then from formal equalities (4) it follows $C = P_2 = \varnothing$. Considering the last condition we have $\Lambda(Q, Q_t) \in Q$ for any $t \in T_m$ and any nonempty element of semilattice Q is a union of some elements of the type $\Lambda(Q, Q_t)$, where $t \in T_m$.

The Theorem is proved.

Theorem 4. *Let* $Q = \{T_1, T_2, T_3, \ldots, T_m\}$ $(m \geq 6)$ *be such subsemilattice of semilattice D, that*

$T_1 \subset T_3 \subset T_4 \subset T_6 \subset \ldots \subset T_m$, $T_1 \subset T_3 \subset T_5 \subset T_6 \subset \ldots \subset T_m$,
$T_2 \subset T_3 \subset T_4 \subset T_6 \subset \ldots \subset T_m$, $T_2 \subset T_3 \subset T_5 \subset T_6 \subset \ldots \subset T_m$,

$T_1 \setminus T_2 \neq \varnothing$, $T_2 \setminus T_1 \neq \varnothing$, $T_4 \setminus T_5 \neq \varnothing$, $T_5 \setminus T_4 \neq \varnothing$, $T_1 \cup T_2 = T_3$, $T_4 \cup T_5 = T_6$

(see Fig. 4). *Then Q is XI-semilattice of unions if and only if* $T_1 \cap T_2 = \varnothing$.

Proof. Let $P_1, P_2, ..., P_{m-1}$ and C be pairwise nonintersected subsets of the set X and φ be the mapping of semilattice Q on the set $\{P_1, P_2, ..., P_{m-1}, C\}$, having the form:

$$\varphi = \begin{pmatrix} T_1 \; T_2 \; ... T_{m-1} \; T_m \\ P_1 \; P_2 \; ... P_{m-1} \; C \end{pmatrix}.$$

Then the formal equalities corresponding to semilattice Q are as follows:

$$\begin{array}{l} T_m = C \cup P_1 \cup P_2 \cup ... \cup P_{m-1}, \quad T_{m-1} = C \cup P_1 \cup P_2 \cup ... \cup P_{m-2}, \\ ------------------------------------ \\ T_6 = C \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5, \quad T_5 = C \cup P_1 \cup P_2 \cup P_3 \cup P_4, \\ T_4 = C \cup P_1 \cup P_2 \cup P_3 \cup P_5, \quad T_3 = C \cup P_1 \cup P_2, \quad T_2 = C \cup P_1, \quad T_1 = C \cup P_2, \end{array} \tag{5}$$

where $|C| \geq 0$, $|P_3| \geq 0$ and $|P_i| \geq 1$ for any i = 1,2,4,5,6,7,..., m–1. Further, let $t \in T_m$. Then from the equalities (5) we get:

$$Q_t = \begin{cases} Q, \; if \;\; t \in C, \\ \{T_2, T_3, ..., T_m\}, \; if \;\; t \in P_1, \\ \{T_1, T_3, ..., T_m\}, \; if \;\; t \in P_2, \\ \{T_4, T_5, ..., T_m\}, \; if \;\; t \in P_3, \\ ------------- \\ \{T_{m-1}, T_m\}, \; if \;\; t \in P_{m-2}, \\ \{T_m\}, \; if \;\; t \in P_{m-1}, \end{cases} \quad \text{and} \quad \Lambda(Q, Q_t) = \begin{cases} T_2, \; if \;\; t \in P_1, \\ T_1, \; if \;\; t \in P_2, \\ T_3, \; if \;\; t \in P_3, \\ -------- \\ T_{m-1}, \; if \;\; t \in P_{m-2}, \\ T_m, \; if \;\; t \in P_{m-1}, \end{cases}$$

We get that $\Lambda(Q, Q_t) \notin Q$, if $t \in C$. Proceeding from Definition 2 at $C \neq \varnothing$ it follows that semilattice Q can not be XI-semilattice.

Further, we assume that $|C| \geq 0$. Thus we consider that $C = \varnothing$. In this case we have $\Lambda(Q, Q_t) \in Q$ for any $t \in T_m$. Besides, it is easily checked that any nonempty element of semilattice Q is a union for some elements of the type $\Lambda(Q, Q_t)$, where $t \in T_m$. Now considering Definition 2, we get that Q is XI-semilattice of unions.

Note that proceeding from the equality (5) condition $C = \varnothing$ is fulfilled if and only if $T_1 \cap T_2 = \varnothing$.

The necessity of conditions of the given theorem is proved and we shall prove the sufficiency of the conditions of the giving theorem. In fact, if $T_1 \cap T_2 = \varnothing$, then from the formal equations (5) it follows $C = \varnothing$. Taking into account the last condition we have $\Lambda(Q, Q_t) \in Q$ for any $t \in T_m$ and any nonempty element of semilattice Q is a union of some elements of the type $\Lambda(Q, Q_t)$, where $t \in T_m$.

The Theorem is proved.

Definition 3. Let $N_m = \{0,1,2,...,m\}$ $(m \geq 1)$ be some subset of the set of all the natural numbers. Subsemilattice $Q = \{T_{ij} \subseteq X \,|\, i \in N_s, \; j \in N_k\}$ of a complete X-semilattice of unions D will be called a net of the size (s+1, k+1), if it contains two subsets $Q_1 = \{T_{00}, T_{10}, ..., T_{s0}\}$, $Q_2 = \{T_{00}, T_{01}, ..., T_{0k}\}$ and satisfies the following conditions:

a) $T_{00} \subset T_{10} \subset ... \subset T_{s0}$ and $T_{00} \subset T_{01} \subset ... \subset T_{0k}$;

b) $Q_1 \cap Q_2 = \{T_{00}\}$;

c) $T_{pq} \neq T_{ij}$, if $(p, q) \neq (i, j)$.

d) elements of the sets Q_1 and Q_2 are pairwise incomparable;

e) $T_{ij} \cup T_{i'j'} = T_{pq}$, if $p = \max\{i, i'\}$ and $q = \max\{j, j'\}$.

Note that the diagram of the net Q is presented in Fig. 5.

Lemma 1. *Let X-*semilattice of unions *Q be the net. Then the following statements are true:*

a) $T_{pq} \subseteq T_{ij}$, *if and only if* $p \le i$ *and* $q \le j$;

b) $Q_1 \cup Q_2$ *is nonreduced generating set of the net Q*;

c) $|Q| = (s+1)\cdot(k+1)$.

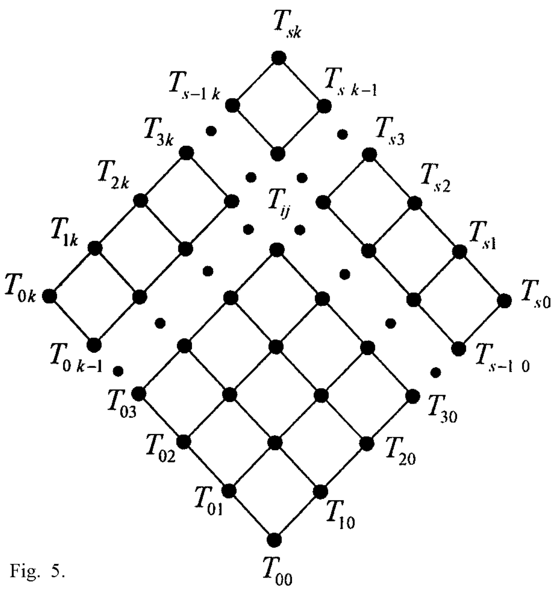

Fig. 5.

Proof. Let $T_{pq}, T_{ij} \in Q$ and $T_{pq} \subseteq T_{ij}$. Then $T_{pq} \cup T_{ij} = T_{ij}$. Proceeding from the conditions e) of Definition 3 we get $i = \max\{p, i\}$ and $j = \max\{q, j\}$, i.e. $p \le i$ and $q \le j$.

On the other hand if for elements $T_{pq}, T_{ij} \in Q$ the conditions $p \le i$ and $q \le j$ are fulfilled, then $\max\{p, i\} = i$, $\max\{q, j\} = j$ and thus $T_{pq} \cup T_{ij} = T_{ij}$. So, the inclusion $T_{pq} \subseteq T_{ij}$ is true.

The statement a) is proved.

Then, let T_{ij} be an arbitrary element of the net Q. Then $0 \le i \le s$, $0 \le j \le k$ on the basis of conditions e) of Definition 3 we get $T_{ij} = T_{i0} \cup T_{0j}$, where $T_{i0} \in Q_1$ and $T_{0j} \in Q_2$. Thus, $Q_1 \cup Q_2$ is a generating set of the net Q.

Now let T_{mn} be such an element of the set $Q_1 \cup Q_2$ and Q' be such own subset of the set $Q_1 \cup Q_2$ that $T_{mn} \notin Q'$ and $T_{mn} = \cup Q'$.

1) If $Q' \subseteq Q_1$ or $Q' \subseteq Q_2$, then by virtue of a) of Definition 3 set Q' is finite chain, therefore $\cup Q'$ coincides with the greatest element of the chain Q'. It follows that $T_{mn} \in Q'$. However, the last condition contradicts the supposition $T_{mn} \notin Q'$.

2) Now let $Q' \cap Q_1 \ne \varnothing$ and $Q' \cap Q_2 \ne \varnothing$. If T_{i0} and T_{0j} are respectively the greatest element of the chain $Q' \cap Q_1$ and $Q' \cap Q_2$, then $T_{mn} = T_{i0} \cup T_{0j}$. Thus by virtue of the condition e) of Definition 3 we get $m = \max\{i, 0\}$ and $n = \max\{0, j\}$, i.e. $m = i$ and $n = j$.

Thus, $T_{i0} = T_{m0}$, $T_{0j} = T_{0n}$ and $T_{mn} = T_{m0} \cup T_{0n}$.

Now taking into account $T_{mn} \in Q_1 \cup Q_2$ we have $m = 0,\ n \ne 0$ or $m \ne 0,\ n = 0$, or $m = n = 0$

From this due to equation $T_{mn} = T_{m0} \cup T_{0n}$ we respectively obtain: $T_{0n} = T_{00} \cup T_{0n} = T_{0n} \in Q_2$ either $T_{m0} = T_{m0} \cup T_{00} = T_{m0} \in Q_1$ or $T_{00} = T_{00} \cup T_{00} = T_{00} \in Q_1$. But conditions $T_{0n} \in Q_2$, $T_{m0} \in Q_1$ and $T_{00} \in Q_1$ contradicts the assumption that $T_{mn} \notin Q'$.

The contradictions obtained in 1) and 2) show that $Q_1 \cup Q_2$ is an irreducible generating set of the net Q.

The statement b) is proved.

The statement c) of the given Lemma follows straight from the conditions c) of the Definition 3.

The Lemma is proved.

Lemma 2. *Let X-semilattice of unions Q be the net. Then the following statements are true:*

a) *formal equations of the net Q have the form* $T_{pq} = P_{sk} \cup \bigcup_{T_{ij} \in Q \setminus Q_{T_{pq}}} \varphi(T_{ij})$, *where* T_{pq} *is an arbitrary element of the net*;

b) *elements of the set* $P_1 = \{P_{0k}, P_{1k}, ..., P_{s-1k}, P_{s0}, P_{s1}, ..., P_{sk-1}\}$ *are basis sources of the net Q*;

c) *elements of the set* $P \setminus P_1$ *are sources of the completeness of the net Q*;

d) $Q^{\wedge} = Q_1 \cup Q_2$.

Proof. First we find a formal equation of the net Q. In fact, let P_{ij} (i=1,2,...,s, j = 1,2,...,k) be some pairwise nonintersecting subsets of the set X and $P=\{P_{ij} | i=1,2,...,s, j = 1,2,...,k\}$. φ is mapping of the semilattice Q on the set P, satisfying the condition $\varphi(T_{ij}) = P_{ij}$ for any $T_{ij} \in Q$. Then the formal equation of the net Q will have the form $T_{pq} = P_{sk} \cup \bigcup_{T_{ij} \in Q \setminus Q_{T_{pq}}} \varphi(T_{ij})$ for any $T_{pq} \in Q$.

The statement a) is proved.

From Figure of the net Q (see Fig. 5) it is seen that the number of the elements covering the elements T_{pk} and T_{sq} (p=0,1,2,…,s-1, q=0,1,2, …, k–1) in the net Q is equal to one. Now we get that elements of the set $P_1 = \{P_{0k}, P_{1k}, ..., P_{s-1k}, P_{s0}, P_{s1}, ..., P_{sk-1}\}$ are basis sources of the net Q.

The statement b) is proved.

If $T_{ij} \in Q \setminus \{T_{0k}, T_{1k}, ..., T_{s-1k}, T_{s0}, T_{s1}, ..., T_{sk-1}\}$ and $T_{ij} \neq T_{sk}$, then element T_{ij} is covered by elements $T_{i+1\,j}$ and $T_{i\,j+1}$ (see Fig. 5). Therefore elements of the set $P \setminus P_1$ are sources of the completeness of the net Q.

The statement c) is proved.

Note that in account of the statement a) of Lemma 1 the following equalities are true:

$$T_{sk} = P_{sk} \cup \bigcup_{T_{ij} \in Q \setminus Q_{T_{sk}}} \varphi(T_{ij}) = P_{sk} \cup \bigcup_{T_{ij} \in Q \setminus \{T_{sk}\}} P_{ij} = \cup P.$$

Therefore for any $t \in \breve{Q}$ there exists such $P_{mn} \in P$ $(0 \le m \le s,\ 0 \le n \le k)$ that $t \in P_{mn}$.

Find the necessary and sufficient conditions when $t \in P_{mn} \subseteq T_{pq}$. In other words for elements $t \in P_{mn}$ we find the necessary and sufficient conditions when $T_{pq} \in Q$.

Really, if $P_{mn} \subseteq T_{pq}$, then, in account of the statement a) of Lemma 1 we shall have that $P_{mn} = P_{sk}$ or $T_{mn} \in Q \setminus Q_{T_{pq}}$.

1) If $P_{mn} = P_{sk}$ then $t \in T_{pq}$ for any $T_{pq} \in Q$. Thus, $Q_t = Q$ and therefore $\Lambda(Q, Q_t) = T_{00}$ by definition of the net Q.

Assume that $(m,n) \neq (s,k)$ and $T_{mn} \in Q \setminus Q_{T_{pq}}$. Relatively to elements m and n we consider the following:

2) $m = s$, $0 \le n < k$. Then in account of $T_{sn} \in Q \setminus Q_{T_{pq}}$ we have $T_{sn} \in Q$ and $T_{sn} \notin Q_{T_{pq}}$. By virtue of the definition of the set $Q_{T_{pq}}$ $(Q_{T_{pq}} = \{T_{ij} \in Q | T_{pq} \subseteq T_{ij}\})$ condition $T_{sn} \notin Q_{T_{pq}}$ takes place if and only if $T_{pq} \nsubseteq T_{sn}$. According to the statement a) of Lemma 1 condition $T_{pq} \nsubseteq T_{sn}$ is fulfilled only when $q \le n$ does not take place as the inequality $p \le s$ is always true by assumption. Thus, the condition $q \le n$ does not take place when $q > n$.

Thus, if $0 \le n < k$ and $T_{sn} \notin Q_{T_{pq}}$, then $q \ge n+1$.

Hence, if $t \in P_{sn}$ and $0 \le n < k$, then $Q_t = \{T_{pq} \in Q | 0 \le p \le s$ and $q \ge n+1$, where $0 \le n < k\}$.

On account of the statement a) of Lemma 1 it follows that T_{0n+1} is the least element of the set Q_t. Therefore,

$\Lambda(Q,Q_t)=T_{0n+1}$ for any $t\in P_{sn}$ and $0\le n<k$.

3) $0\le m<s$, $n=k$. Then due to $T_{mk}\in Q\setminus Q_{T_{pq}}$ we have $T_{mk}\in Q$ and $T_{mk}\notin Q_{T_{pq}}$. By virtue of the definition of the set $Q_{T_{pq}}$, condition $T_{mk}\notin Q_{T_{pq}}$ takes place only when $T_{pq}\not\subseteq T_{mk}$. On account of the statement a) of Lemma 1 condition $T_{pq}\not\subseteq T_{mk}$ is fulfilled only when $p\le m$ does not take place as the condition $q\le k$ is always true by assumption. Thus, the condition $p\le m$ does not take place when $p>m\ge 0$.

Thus, if $0\le m<s$ and $T_{mk}\notin Q_{T_{pq}}$, then $p\ge m+1$.

Hence, if $t\in P_{mk}$ and $0\le m<s$, then $Q_t=\{T_{pq}\in Q\mid 0\le p\le k$ and $q\ge m+1$, where $0\le m<s\}$.

On account of the statement a) of Lemma 1 it follows that T_{m+10} is the least element of the set Q_t. Therefore $\Lambda(Q,Q_t)=T_{m+10}$ for any $t\in P_{mk}$ and $0\le m<s$.

4) $0\le m<s$, $0\le n<k$ and $T_{mn}\in Q\setminus Q_{T_{pq}}$. Then $T_{mn}\in Q$ and $T_{mn}\notin Q_{T_{pq}}$. By virtue of the definition of the set $Q_{T_{pq}}$ condition $T_{mn}\notin Q_{T_{pq}}$ takes place only when $T_{pq}\not\subseteq T_{mn}$. On account of the statement a) of Lemma 1 condition $T_{pq}\not\subseteq T_{mn}$ is fulfilled only when one of the conditions $p\le m$ or $q\le n$ does not take place. Conditions $p\le m$ or $q\le n$ do not take place when $p>m\ge 0$ or $q>n\ge 0$.

Thus, if $0\le m<s$, $0\le n<k$ and $T_{mn}\notin Q_{T_{pq}}$, then $p\ge m+1$ or $q\ge n+1$.

Hence, if $t\in P_{mn}$ and $0\le m<s$, $0\le n<k$, then $Q_t=\{p\ge m+1,\ q\ge n+1$, where $0\le m<s$ and $0\le n<k\}$.

If $t\in P_{mn}$ and $0\le m<s$, $0\le n<k$, then $T_{m+10},\ T_{0n+1}\in Q_t$. From this and by definition of the net Q we obtain $\Lambda(Q,Q_t)=T_{00}$.

Now taking into account the results obtained in items 1)-4) we have

$Q^{\wedge}=\{T_{00},T_{01},\ldots,T_{0k},T_{10},T_{10},\ldots,T_{s0}\}=Q_1\cup Q_2$.

The statement d) is proved.

Lemma is proved.

From the statement b) of Lemma 1 and from definition of basis sources of semilattice Q it follows that for the existence of the net Q size (s+1, k+1) it is necessary $|X|\ge s+k$. In these conditions we have $|B_X(Q)|\ge|Q|^{s+k}$.

Theorem 5. *If Q is a net, then it is XI-semilattice of unions.*

Proof. First note that $N(Q,Q_t)\ne\varnothing$ ($T_{00}\in N(Q,Q_t)$) for any $t\in\breve{Q}$.

From here and in account of the statement a) of Lemma 1 we obtain $\Lambda(Q,Q_t)\in Q$ for any $t\in\breve{Q}$.

On account of the statement d) of Lemma 1 we have $Q^{\wedge}=Q_1\cup Q_2$. Besides, on account of the statement b) of Lemma 1 it follows that $Q^{\wedge}$ is a generating set of the net Q. So any nonempty element of the set Q is a union of some elements of the set $Q^{\wedge}$. From this and taking into account the fact that $\Lambda(Q,Q_t)\in Q$ for any $t\in\breve{Q}$ we obtain that Q is XI-semilattice of unions.

The Theorem is proved.

მათემატიკა

გაერთიანებათა XI-ნახევარმესერების შესახებ

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სტატიაში აღწერილია გაერთიანებათა ზოგიერთი XI-ნახევარმესერი. ისინი გამოიყენება ბინარულ მიმართებათა ისეთი სრული ნახევარჯგუფების შესწავლისას, რომლებიც განსაზღვრულნი არიან გაერთიანებათა სრული X-ნახევარმესერებით. მოცემულ გაერთიანებათა სრული X-ნახევარმესერის ყველა XI-ქვენახევარმესერთა ცოდნა კი საშუალებას იძლევა აღწერილ იქნას მოცემული ნახევარჯგუფის ყველა იდემპოტენტი და რეგულარული ელემენტები.

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