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Mathematics

Homology and Cohomology Groups of Group Homomorphism

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ABSTRACT. The Japanese mathematician S.Takasu first constructed homology and cohomology groups of the pair (*G*, *H*) of a group and a subgroup by using the embedding $f : H \to G$. In the present paper, this theory is generalized for any homomorphism of groups and a number of classic results are proved. Cohomology group of the second order is characterized by the use of relative widening. *© 2008 Bull. Georg. Natl. Acad. Sci*

Key words: *groups of homologies and cohomologies.*

As is known, groups of homologies and cohomologies of Massey-Takasu for group pairs (G,H) are defined by embedding $f: H\to G, H<G$ [1].

In our case these definitions are generalized for arbitrary group homomorphism $\varphi: \Pi\to G$. In particular, homology and cohomology groups of n order for the homomorphism φ are defined and if $\Pi=H$, we get the known theory [2].

Let A be an arbitrary G-module. Then A gets Π-module structure in a natural way. If we consider G-projective resolvent $X\to Z$ and Π-projective resolvent $Y\to Z$ (Z is considered to be trivial G-module) over the ring of integral numbers Z, then there exists a chain transformation $F:Y\to X$ which generates the mapping $\overline{F}: Y \underset{\Pi}{\overset{\varphi}{\otimes}} A \to X \underset{G}{\otimes} A$, $\overline{F}(y\otimes a) = F(y)\otimes a$.

Let us construct the $\overline{F}$ -relative chain complex $N_*(\overline{F}) = \sum_n N_n(\overline{F})$ [1].

Definition 1. Let the homology group of n order over the constructed complex $N_*(\overline{F})$ be called the homology group of n order of homomorphism φ and denote it as follows:

$$H_n(\varphi, A) = H_n(N_*(\overline{F})).$$

Let us consider G-projective resolvent $X\to Z$ and Π-projective resolvent $Y\to Z$ over Z. In this case there exists $E:Y\to X$ which, in its turn, causes mapping $\overline{E}: Hom_G(X,A) \to Hom_\Pi(Y,A)$, $\overline{E}(\alpha) = \alpha\cdot E$, $\alpha\in Hom_G(X,A)$. Let us construct the $\overline{E}$ -relative cochain complex $N^*(\overline{E}) = \sum_n N^n(\overline{E})$ [2].

Definition 2. Let the homology group of n order over constructed complex $N^*(\overline{E})$ be called the cohomology group of n order of homomorphism φ and denote it as:

$$H^n(\varphi, A) = H_n\left(N^*\left(\overline{E}\right)\right)$$

Definition 3. Tensor product $A \overset{\varphi}{\underset{\Pi}{\otimes}} G$ is called Abelian group generated by symbols $a \otimes g$, where $a \in A$, $g \in G$ which are interrelated in the form:

$$(a+a') \otimes g = a \otimes g + a' \otimes g\,,\ a \otimes (g+g') = a \otimes g + a \otimes g'\,,$$

$$a \otimes x \cdot g = a \cdot x \otimes g\,,\ a, a' \in A\,,\ g, g' \in G\,,\ z \in \Pi\,,$$

$$x \cdot g = \varphi(x) \cdot g\,,\ ax = a \cdot \varphi(x)\,.$$

Define the group $Hom_{\Pi}^{\varphi}(G, A)$ as the set of all the $\varphi(\Pi)$-homomorphisms from G into A, i.e.

$$Hom_{\Pi}^{\varphi}(G, A) = \left\{f : G \to A \,/\, f\left[\varphi(x) \cdot g\right] = \varphi(x) \cdot f(g)\right\}.$$

The constructed group receives G-module structure in the following way: for arbitrary $g, g' \in G$

$$(g \cdot f)(g') = f(g' \cdot g).$$

Let X and X' be G-projective and Π-projective resolvents respectively over Z. Then for identical mapping over Z there exists mapping $F : X' \to X$, which, in its turn, assumes

$$\overline{F} : X' \overset{\varphi}{\underset{\Pi}{\otimes}} G \to X\,,\ \overline{F}(x' \otimes g) = F(x') \cdot g\,,\ x' \in X\,,\ g \in G\,.$$

Define the cylinder of algebraic mapping $M^Z\left(\overline{F}\right)$ and $\overline{F}$-relative chain complex $N^Z\left(\overline{F}\right)$ as follows:

$$\begin{cases} M_0(\overline{F}) = \left(X_0' \overset{\varphi}{\underset{\Pi}{\otimes}} G\right) \otimes X_0, \\ \partial_0(x_0' \otimes g, x_0) = 0, \quad (x_0' \otimes g, x_0) \in M_0(\overline{F}), \\ M_n(\overline{F}) = \left(X_n' \overset{\varphi}{\underset{\Pi}{\otimes}} G\right) \otimes \left(X_{n-1}' \overset{\varphi}{\underset{\Pi}{\otimes}} G\right) \otimes X_n, \quad n \geq 1, \\ \partial_n(x_n' \otimes g, x_{n-1}' \otimes g', x_n) = ((\partial x_n') \otimes g - x_{n-1}' \otimes g', -(\partial x_{n-1}') \otimes g, \partial x_n + \overline{F}(x_{n-1}' \otimes g')) \\ (x_n' \otimes g, x_{n-1}' \otimes g', x_n) \in M_n(\overline{F}), \quad n \geq 2, \\ \partial_1\left(X_1' \otimes g, x_0' \otimes g', x_1\right) = \left((\partial x_1') \otimes g - x_0' \otimes g', -\partial x_1 + \overline{F}(x_0' \otimes g')\right) \end{cases}$$

$$\begin{cases} N_0(\overline{F}) = X_0, \quad \partial_0 X_0 = 0, \\ N_n(\overline{F}) = \left(X_{n-1}' \overset{\varphi}{\underset{\Pi}{\otimes}} G\right) \otimes X_n, \quad n \geq 1, \\ \partial_n(x_{n-1}' \otimes g, x_n) = (-(\partial x_{n-1}') \otimes g, \partial x_n + \overline{F}(x_{n-1}' \otimes g)), \quad n \geq 1, \\ \partial_1(x_0' \otimes g, x_1) = \partial x_1 + \overline{F}(x_0' \otimes g). \end{cases}$$

Theorem 1. *Cylinder of algebraic mapping $M^Z\left(\overline{F}\right)$ for $\overline{F}$ is G-projective resolvent over Z.*

Proof. Construct the mapping

$$\overline{\varepsilon} : M_0\left(\overline{F}\right) \to Z\,,\ \overline{\varepsilon}\left(x_0' \otimes g, x_0 = \varepsilon'\left(x_0'\right) \cdot g + \varepsilon\left(x_0\right)\right),$$

where $\varepsilon' : X_0' \to Z$ and $\varepsilon : X_0 \to Z$ are given mappings. Then

$$\bar{\varepsilon}\partial_1\left(x_1' \otimes g, x_0' \otimes g', x_1\right) = -\varepsilon'\left(x_0'\right)\cdot g + \varepsilon\left(\overline{F}\left(x_0' \otimes g'\right)\right) = -\varepsilon'\left(x_0'\right)\cdot g + \varepsilon'\left(x_0'\right)\cdot g = 0 .$$

Besides, $Ker\left(\bar{\varepsilon}\right) \subset \operatorname{Im}\partial_1$ i.e. if $\left(x_0' \otimes g, x_0\right) \in Ker\left(\bar{\varepsilon}\right)$, then $F\left(x_0'\right)\cdot g + x_0 \in Ker(\varepsilon)$ as

$$\bar{\varepsilon}\left(x_0' \otimes g, x_0\right) = \varepsilon'\left(x_0'\right)\cdot g + \varepsilon\left(x_0\right) = \varepsilon\left(F\left(x_0'\right)\cdot g\right) + \varepsilon\left(x_0\right).$$

From acyclicity X it follows that there exists $x_1 \in X_1$ for which $\partial_1\left(x_1\right) = F\left(x_0'\right)\cdot g + x_0$. As $\partial_1\left(0, -x_0' \otimes g, x_1\right) = \left(x_0' \otimes g, x_0\right)$, then $\left(x_0' \otimes g, x_0\right) \in \operatorname{Im}\partial_1$. Thus, $\operatorname{Im}\partial_{n+1} = Ker\partial_n$, $n \geq 1$. Therefore, $H_n\left(M^Z\left(\overline{F}\right)\right) = H_n(X)$, $n \geq 0$. Hence, $M^Z\left(\overline{F}\right)$ is G-projective resolvent over Z.

Theorem 2. (1). *For arbitrary G-module A,* $N^Z\left(\overline{F}\right)\underset{G}{\otimes} A$ *is relative chain complex mapping*

$$\Phi : \left(X' \overset{\varphi}{\underset{\Pi}{\otimes}} G\right)\underset{G}{\otimes} A \to X \underset{G}{\otimes} A$$

and the following equality takes place:

$$H_n(\varphi, A) = H_n\left(N^Z\left(\overline{F}\right)\underset{G}{\otimes} A\right);$$

(2). *If Y is G-projective resolvent over A, then* $N^Z\left(\overline{F}\right)\underset{G}{\otimes} Y$ *is relative chain complex mapping:*

$$\Phi' : \left(X' \overset{\varphi}{\underset{\Pi}{\otimes}} G\right)\underset{G}{\otimes} Y \to X \underset{G}{\otimes} Y$$

and the equality takes place:

$$H_n(\varphi, A) = H_n\left(N^Z\left(\overline{F}\right)\underset{G}{\otimes} Y\right).$$

Proof. (1) is evident;

(2).
$$\sum_{i+j=n-1}\left(\left(X_i' \overset{\varphi}{\underset{\Pi}{\otimes}} G\right)\underset{G}{\otimes} Y_j\right) + \sum_{i+j=n} X_j \underset{G}{\otimes} Y_j =$$

$$= X_0 \underset{G}{\otimes} Y_n + \left(X_0' \overset{\varphi}{\underset{\Pi}{\otimes}} G + X_1\right)\underset{G}{\otimes} Y_{n-1} + \cdots + \left(X_{n-1}' \underset{\Pi}{\otimes} G + X_n\right)\underset{G}{\otimes} Y_0 = \sum_{i+j=n} N_i\left(\overline{F}\right)\underset{G}{\otimes} Y_j .$$

Theorem 3. *For each short exact sequence of G-modules and G-module homomorphisms*

$$0 \to A \to B \to C \to 0$$

there exists a long exact homology sequence:

$$\cdots \to H_n(\varphi, A) \to H_n(\varphi, B) \to H_n(\varphi, C) \to H_{n-1}(\varphi, A) \to \cdots$$

Proof. It is evident that there exist G-projective resolvents X, Y and T respectively over A, B, and C, for which diagram

$$\begin{array}{ccccccccc} 0 & \to & X & \to & Y & \to & T & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & B & \to & C & \to & 0 \end{array}$$

is commutative and the rows are exact. As $N^Z(\overline{F})$ is G-projective resolvent, then the sequence

$$0 \to N^Z(\overline{F}) \underset{G}{\otimes} X \to N^Z(\overline{F}) \underset{G}{\otimes} Y \to N^Z(\overline{F}) \underset{G}{\otimes} T \to 0$$

is exact. According to Theorem 2 (2) there exists the desired exact sequence.

Theorem 4. *If G is Π–projective module, then*

$$H_n(\varphi, A) \cong Tor_{n-1}^G\left(I_{(G,\Pi)}^{\varphi}(Z), A\right), \quad n \geq 1,$$

where

$$I_{(G,\Pi)}^{\varphi}(Z) = Ker\left(Z \overset{\varphi}{\underset{\Pi}{\otimes}} G \to Z\right).$$

Proof. According to the definition the sequence

$$0 \to X' \overset{\varphi}{\underset{\Pi}{\otimes}} G \to M^Z(\overline{F}) \to N^Z(\overline{F}) \to 0$$

is exact and a corresponding homology sequence gives an isomorphism:

$$H_1\left(N^Z(\overline{F})\right) \cong Ker\left(Z \overset{\varphi}{\underset{\Pi}{\otimes}} G \to Z\right)$$

and

$$H_n\left(N^Z(\overline{F})\right) \cong Tor_{n-1}^{\Pi}(G, Z), \quad n \geq 2.$$

If G is Π-projective module, then

$$H_n\left(N^Z(\overline{F})\right) = \begin{cases} I_{(G,\Pi)}^{\varphi}(Z), & n = 1 \\ 0, & n \neq 1 \end{cases}$$

Let us construct a chain complex $\overline{N}$ in the following way:

$$\overline{N}_0 = 0, \ \overline{N}_1 = Ker\left(N_1^Z(\overline{F}) \to N_0^Z(\overline{F})\right), \ \overline{N}_n = N_n^Z(\overline{F}), \ n \geq 2.$$

A boundary operator in $\overline{N}$ will be defined by analogy with the complex $N^Z(\overline{F})$.

Define the mapping

$$\overline{\varepsilon} : \overline{N}_1 \to Z \overset{\varphi}{\underset{\Pi}{\otimes}} G, \quad \overline{\varepsilon}(x_0 \otimes g, x_1) = \varepsilon'(x_0') \otimes g,$$

where $\varepsilon' : X' \to X$ is the given mapping.

To prove Theorem 4 it is enough to prove that:

(1) $\mathrm{Im}(\overline{\varepsilon}) = I_{(G,\Pi)}^{\varphi}(Z)$;

(2) $\mathrm{Im}(\partial_2) = Ker(\overline{\varepsilon})$;

(3) $\overline{N}_1$ is G-projective module;

(4) $H_n\left(N^Z\left(\overline{F}\right)\underset{G}{\otimes}A\right)\cong H_n\left(\overline{N}\underset{G}{\otimes}A\right)$.

Really, (1). Consider the commutative diagram

$$\begin{array}{ccccccccc} \overline{N}_2 & \to & \overline{N}_1 & \xrightarrow{i} & N_1 & \xrightarrow{p} & X_0'\overset{\varphi}{\underset{\Pi}{\otimes}}G & \xrightarrow{\varepsilon'} & Z\overset{\varphi}{\underset{\Pi}{\otimes}}G \\ & & & & & & \Big|\overline{F} & & \Big|\alpha \\ & & & & X_1 & \to & X_0 & \to & Z \end{array}$$

As $\partial x_1+\overline{F}(x_0'\otimes g)=0$, where $\left(x_0'\otimes g,x_1\right)\in\overline{N}_1=Ker\partial_1$ then

$$\alpha\overline{\varepsilon}(x_0'\otimes g,x_1)=\alpha\left(\varepsilon'(x_0')\otimes g\right)=\varepsilon'(x_0')\cdot g=\varepsilon\overline{F}(x_0'\otimes g)=-\varepsilon\partial x_1=0.$$

Hence, $\operatorname{Im}\left(\overline{\varepsilon}\right)\subset I^{\varphi}_{(G,\Pi)}(Z)$. On the contrary, if $\alpha(n\otimes g)=0,$ then there exists an element $x_0'\otimes g\in X_0'\overset{\varphi}{\underset{\Pi}{\otimes}}G$ for which $\varepsilon'(x_0')\otimes g=n\otimes g$. As $\varepsilon\overline{F}(x_0'\otimes g)=0$ then there exists $x_1\in X_1$ for which $\partial x_1+\overline{F}(x_0'\otimes g)=0$

Thus,

$$\overline{\varepsilon}(x_0'\otimes g,x_1)=\varepsilon'(x_0')\otimes g=n\otimes g,$$

i.e. $\operatorname{Im}\left(\overline{\varepsilon}\right)=I^{\varphi}_{(G,\Pi)}(Z)$.

(2) As $\overline{\varepsilon}\partial_2(x_1'\otimes g,x_2)=-\left(\varepsilon'\partial x_1'\right)\otimes g=0$, then $\operatorname{Im}\partial_2\subset Ker\left(\overline{\varepsilon}\right)$.

On the contrary, if $\overline{\varepsilon}(x_0'\otimes g,x_1)=-\left(\varepsilon' x_0'\right)\otimes g=0,$ then from Π–projectivity G there exists an element $x_1'\otimes g\in X_1'\overset{\varphi}{\underset{\Pi}{\otimes}}G$ for which $-\partial x_1'\otimes g=x_0'\otimes g$. As $(x_0'\otimes g,x_1)\in\overline{N}_1$, then $\partial x_1=-\overline{F}(x_0'\otimes g)=\partial\overline{F}(x_1'\otimes g)$. Hence, there exists an element $x_2\in X_2$, for which $\partial x_2=x_1-\overline{F}(x_1'\otimes g)$. Then

$$\partial_2(x_1'\otimes g,x_2)=\left(-\partial x_1'\otimes g,\partial x_2+\overline{F}(x_1'\otimes g)\right)=(x_0'\otimes g,x_1).$$

Thus, $\operatorname{Im}\partial_2=Ker\left(\overline{\varepsilon}\right)$.

(3) For an arbitrary element $x_0\in X$ there exists an element $n\otimes g\in Z\overset{\varphi}{\underset{\Pi}{\otimes}}G$ for which $\varepsilon(x_0)=\alpha\left(n\underset{\varphi}{\otimes}g\right)$. Also, there exists an element $x_0'\otimes g'\in X_0'\overset{\varphi}{\underset{\Pi}{\otimes}}G$ for which $\varepsilon'(x_0')\otimes g'=n\otimes g$. As $\varepsilon\left(\overline{F}(x_0'\otimes g')-x_0\right)=0,$ then there exists $x_1\in X_1$ for which $-\partial x_1=\overline{F}(x_0\otimes g')-x_0$.

Hence, $x_0=\partial x_1=\overline{F}(x_0'\otimes g')\in\operatorname{Im}(\partial_1)$. Thus, the sequence

$$0\to Ker(\partial_1)\to X_0'\overset{\varphi}{\underset{\Pi}{\otimes}}G\to X_0\to 0$$

is exact. As X_0 is G-projective, then this sequence is released, i.e. $Ker(\partial_1)$ is G-projective

(4) Construct a chain complex $-\overline{\overline{N}}=N\left(\overline{F}\right)/\overline{N}$ in the following way:

$$\overline{\overline{N}}_n = 0\,,\; n \geq 2\,;\;\; \partial_n = 0\,,\; n \geq 2\,;\;\; \overline{\overline{N}}_1 = N_1\left(\overline{F}\right)/\overline{N}_1\,,\;\; \overline{\overline{N}}_0 = N_0\,.$$

For boundary operator ∂_1 we get an isomorphism $\overline{\overline{N}}_1 \approx \overline{\overline{N}}_0$. Hence, $H_n\left(\overline{\overline{N}} \underset{G}{\otimes} A\right) = 0$. On the other hand, from (3) it follows that the sequence

$$0 \to \overline{N} \to N \to \overline{\overline{N}} \to 0$$

is decomposable and we get an exact sequence:

$$0 \to \overline{N} \underset{G}{\otimes} A \to N \underset{G}{\otimes} A \to \overline{\overline{N}} \underset{G}{\otimes} A \to 0\,.$$

From the sequence of homology of groups we get

$$H_n\left(\overline{\overline{N}} \underset{G}{\otimes} A\right) = 0\,.$$

Thus, an isomorphism takes place:

$$H_n\left(\overline{N} \underset{G}{\otimes} A\right) \cong H_n\left(N^Z\left(\overline{F}\right) \underset{G}{\otimes} A\right).$$

Corollary. *If* Π=*H is a subgroup in G, then*

$$H_n(\varphi, A) = H_n(G, H, A)\,,$$

where $H_n(G,H,A)$ is relative to the group of homologies of n order of Massey-Takasu [2].

Proof. It is evident that if $H<G$, then

$$Z[G] = \underset{G/H}{\otimes} Z[H]\,,$$

i.e. $Z[G]$ is a free $Z[H]$-module and, moreover, $Z[H]$-projective module.

Theorem 5. *If G is a* Π-*projective module, then*

$$H^n(\varphi, A) \cong Ext_G^{n-1}\left(I^{\varphi}_{(G,\Pi)}(Z), A\right),\; n \geq 1\,.$$

Proof. Analogously to Theorem 4.

Corollary. *If* Π=*H is a subgroup in G, then*

$$H^n(\varphi, A) = H^n(G, H, A)\,,$$

where $H^n(G,H,A)$ is relative to the group of cohomologies of n order of Massey-Takasu [2].

We should note that for G-module A it is possible to construct φ-relative injective resolvent in a dual way. For this, let us consider G-injective and Π-injective resolvents Y and Y' over A. Then there exists mapping $E: Y \to Y'$. Let

$$N^0\left(\overline{E}\right) = Y^0\,,\;\; \delta(y) = \left(E(y), \delta(y)\right),$$

$$N^n\left(\overline{E}\right) = Hom^{\varphi}_{\Pi}\left(G, y'^{n-1}\right) \otimes Y^n\,,$$

$$\delta(\alpha', y) = \left(-\partial\alpha' + \overline{E}(y), \partial(y)\right), (\alpha', y) \in N^n\left(\overline{E}\right),$$

where $\overline{E}: Y \to Y' = Hom^{\varphi}_{\Pi}\left(G, Y'\right)$ is G-mapping, which is defined as: $\overline{E}(y)(g) = E(g \cdot y)$. Then the following Theorem is true.

Theorem 6. *If G is a Π-projective module, then*

$$H^n(\varphi, A) \cong Ext_G^{n-1}\left(Z, j_{(G,\Pi)}^{\varphi}(A)\right),$$

where $j_{(G,\Pi)}^{\varphi}(A) = Co\ker\left(A \to Hom_{\Pi}^{\varphi}(G, A)\right)$.

Corollary. *If Π=H is a subgroup in G, then*

$$H^n(\varphi, A) = H^n\left(G, H, j_{(G,\Pi)}^{\varphi}(A)\right).$$

Let X and X' G-projective and Π-projective resolvents respectively over Z. Construct a cochain complex N in the following way:

$$N^n = Hom_{\Pi}\left(X'_{n-1}, A\right) \otimes Hom_G\left(X_n, A\right),$$

$$\delta(\alpha, \beta) = \left(-\partial'\alpha + \mathrm{Re}\,sg, \delta\beta\right), \quad (\alpha, \beta) \in N^n.$$

It is easy to check that $H^n(\varphi, A) = H^n(N)$. $(\alpha, \beta) \in Z^2(N)$ is a two-dimensional cocycle if and only if the equality takes place:

$$g_1\beta(g_2, g_3) + \beta(g_1, g_2g_3) = \beta(g_1, g_2) + \beta(g_1g_2, g_3),$$

where $g_1, g_2, g_3 \in G$ and

$$\mathrm{Res}\,(x, x') = x\alpha(x') - \alpha(xx') + \alpha(x),$$

where $x, x' \in \Pi$.

$(\alpha, \beta) \in B^2(N)$ is coboundary if and only if the following conditions are fulfilled:

$$\beta(g_1, g_2) = g_1\beta(g_2) - h(g_1g_2) + h(g_1), \quad g_1, g_2 \in G,$$

$$\alpha(x) = x\alpha - a + \mathrm{Re}\,sh(x), \quad a \in A, \quad x \in \Pi$$

and h is a fixed element from $Hom_G(X_1, A)$, $a \in A$ is also a fixed element.

Definition 4. For G-module A φ-widening is called the three $\left(\overline{G}, \overline{\Pi}, p\right)$, for which the following condition is true:

(1) G is a multplicative group, in which A is a subgroup;

(2) $p : \overline{G} \to G$ is epimorphism, for which $Ker(p)$=A (i.e. p isomorphism $i : \overline{G}/A \cong G$);

(3) $eae^{-1} = p(e)$, where $e \in \overline{G}$, and $p(e)$ is the right G-operator over A;

(4) $\overline{\Pi}$ is a subgroup in $\overline{G}$, for which $p|_{\overline{\Pi}} : \overline{\Pi} \cong \Pi$.

Definition 5. For G-module A, φ-widened $\left(\overline{G}_1, \overline{\Pi}_1, p_1\right)$ and $\left(\overline{G}_2, \overline{\Pi}_2, p_2\right)$ are called equivalent, if there exists a group isomorphism $t : \overline{G}_1 \cong \overline{G}_2$, for which $t|_A$ is identical, $t|_{\overline{\Pi}} : \overline{\Pi}_1 \cong \overline{\Pi}_2$ and $p_2 \cdot t = p_1$.

Let $\left(\overline{G}, \overline{\Pi}, p\right)$ be φ-widening of G-module A. Consider a complete system of representatives of cosets in the factor-group $\overline{G}/A$ and define mapping $q : G \to \overline{G}$ which represents an element $g \in G$ in $i^{-1}(g) \in \overline{G}/A$. Evidently, $p \cdot q = 1_G$ and $g \cdot a = q(g) \cdot a \cdot q(g)^{-1}$, $g \in G$, $a \in A$. Define cocycle $\beta \in Z^2(G, A)$ as follows:

$$q(g_1) \cdot q(g_2) = \beta(g_1, g_2) \cdot q(g_1, g_2), \quad g_1, g_2 \in G, \quad \beta\left(g_1, g_2\right) \in A,$$

$$\beta(g_1, g_2) + \beta(g_1g_2, g_3) = g_1\beta(g_2, g_3) + \beta(g_1, g_2g_3), \quad g_1, g_2, g_3 \in G.$$

As $(\overline{G},\overline{\Pi},p)$ is φ-widening of G-module A, therefore for each element $x\in\Pi$ there exists $\alpha(x)\in A$ for which $q(x)=\alpha(x)\cdot\overline{x}$, where $x=p(\overline{x})$. From (1) it follows that a and b are interrelated in the following way:

$$\beta(x,x')=\alpha(x)+x\cdot\alpha(x')-\alpha(xx'),\ x,x'\in\Pi.$$

If we consider another system of representatives $r:G\to G'$ and define a cocycle $\overline{\beta}\in Z^2(G,A)$ in a similar way over the definite cocycle $\beta\in Z^2(G,A)$, then cocycles β and β' will be interrelated in the following way:

$$\overline{\beta}(g_1,g_2)=\beta(g_1,g_2)+h(g_1)+g_1h(g_2)-h(g_1\cdot g_2), \tag{2}$$

$g_1,g_2\in G$ and h is defined by the equality: $r(g)=h(g)\cdot q(g)$, $g\in G$ and $h(g)\in A$. If $t:(\overline{G}_1,\overline{\Pi}_1,p_1)\approx(\overline{G}_2,\overline{\Pi}_2,p_2)$ is equivalency of φ-widenings of G-module A, and $q:G\to\overline{G}_1$ and $r:G\to\overline{G}_2$, then β and $\overline{\beta}$, which are defined according to mapping q and r, must satisfy the equality (2), where h is defined by the following condition: $r(g)=h(g)\cdot t(q(g))$. On the other hand, for φ-widening $(\overline{G},\overline{\Pi},p)$ of G-module A, an equivalent φ-widening $(\overline{G}',\overline{\Pi}',p')$ is defined which satisfies the condition: $\overline{\Pi}'=a^{-1}\overline{\Pi}a$ for each fixed element $a\in A$.

Let $t:(\overline{G}_1,\overline{\Pi}_1,p_1)\approx(\overline{G}_2,\overline{\Pi}_2,p_2)$ is an equivalency of φ-widenings of G-module A. Then

$$t\cdot q(x)=\alpha(x)\cdot t(\overline{x})=\alpha(x)\cdot a^{-1}\cdot\overline{x}\cdot a=\alpha(x)a^{-1}\overline{x}a\overline{x}^{-1}\overline{x}=\alpha(x)a^{-1}(xa)\overline{x},$$

i.e. there exists α in $(\overline{G}_2,\overline{\Pi}_2,p_2)$ for which

$$\overline{\alpha}(x)=\alpha(x)-a+xa,\ x\in\Pi.$$

But such change of α does not influence cocycle β, i.e. $\beta(x,x')=\overline{\beta}(x,x')$.

Thus, the class of equivalency of each φ-widening of G-module A unambiguously defines the cohomology class in $H^2(\varphi,A)$. On the contrary, it is possible to construct for $(\alpha,\beta)\in Z^2(\varphi,A)$ φ-widening $(\overline{G},\overline{\Pi},p)$ and if (α,β) and (α',β') are cohomologic, then we get an equivalent φ–widening $(\overline{G}_1,\overline{\Pi}_1,p_1)$ and $(\overline{G}_2,\overline{\Pi}_2,p_2)$.

Hence we conclude that the following theorem is true.

Theorem 7. *Between the group $H^2(\varphi,A)$ and the set of equivalent classes of φ-widenings of G-module A there exists mutual unambiguous correlation.*

მათემატიკა

ჯგუფური ჰომომორფიზმის ჰომოლოგიური და კოჰომოლოგიური ჯგუფები

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ცნობილი იაპონელი მათემატიკოსის ს. ტაკასუს მიერ პირველად იქნა აგებული ჯგუფისა და ქვეჯგუფის (G, H) წყვილის ჰომოლოგიური და კოჰომოლოგიური ჯგუფები $f : H \to G$ ჩადგმის გამოყენებით. წინამდებარე ნაშრომში განზოგადებულია ეს თეორია ჯგუფთა ნებისმიერი ჰომომორფიზმის შემთხვევაში და მისთვის დამტკიცებულ იქნა მთელი რიგი კლასიკური შედეგები. აქვეა მოცემული მეორე რიგის კოჰომოლოგიერი ჯგუფის დახასიათება ფარდობითი გაფართოებების საშუალებით.

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