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Mathematics

On the Power of the Goodness-of-Fit Test Based on Wolverton-Wagner Distribution Density Estimates

Elizbar Nadaraya*, **Petre Babilua****

* *Academy Member, I. Javakhishvili Tbilisi State University*
** *I. Javakhishvili Tbilisi State University*

ABSTRACT. A goodness-of-fit test is constructed by using a Wolverton-Wagner distribution density estimate. The question as to its consistency is studied. The power asymptotics of the constructed goodness-of-fit test is also studied for certain types of close alternatives. *© 2008 Bull. Georg. Natl. Acad. Sci.*

Key words: *Distribution density, goodness-of-fit test, power, consistency, limit distribution.*

1. Let $X_1, X_2, \ldots, X_n$ be a sequence of independent, equally distributed random variables with values in a Euclidean p-dimensional space R_p, $p \ge 1$, which have the distribution density $f(x)$, $x = (x_1, \ldots, x_p)$. Using the sampling $X_1, X_2, \ldots, X_n$, it is required to test the hypothesis

$$H_0 : f = f_0 .$$

We consider the verification test of the hypothesis H_0, based on the statistics

$$U_n = n a_n^{-p} \int \left(f_n(x) - f_0(x)\right)^2 r(x)\, dx ,$$

where $f_n(x)$ is a kernel estimate of the Wolverton-Wagner probability density,

$$f_n(x) = \frac{1}{n} \sum_{i=1}^{n} a_i^p K\left(a_i (x - X_i)\right),$$

$a_1, a_2, \ldots, a_n$ is a sequence of positive numbers monotonically converging to infinity.

Let us formulate the conjectures concerning $K(x)$ and $f(x)$.

1⁰. The kernel $K(x) = \prod_{j=1}^{p} K_j(x_j)$ and every kernel $K_j(u)$ possesses the following properties:

$$0 \le K_j(u) \le c < \infty, \quad K_j(u) = K_j(-u), \quad u^2 K_j(u) \in L_1(-\infty, \infty),$$
$$\int K_j(u)\, du = 1, \quad K_j^0(ux) \ge K_j^0(x) \quad \text{for all} \quad u \in [0,1] \quad \text{and all} \quad x \in R_1 = (-\infty, \infty),$$

where $K_j^0 = K_j * K_j$; $*$ is the convolution operator.

2⁰. The distribution density $f(x)$ is bounded and has bounded partial derivatives up to second order.

We have proved the following theorem ([1]-[3]).

Theorem 1. *Let* $K(x)$ *and* $f(x)$ *satisfy conjectures* 1^0 *and* 2^0, *respectively, and, besides, let the second order partial derivatives of the function* $f(x)$ *belong to* $L_1(R_p)$. *If*

$$\frac{a_n^p}{n} \to 0, \quad \frac{\gamma_s(n)}{a_n^{sp}} \to \gamma_s, \quad s = 1,2 \quad (0 < \gamma_2 \le \gamma_1 \le 1)$$

and also

$$\frac{\gamma_1(n)}{a_n^p} = \gamma_1 + o\left(a_n^{-p/2}\right), \quad \left(na_n^{p/2}\right)^{-1} \sum_{k=1}^{n} a_k^{p-2} \to 0$$

and

$$\left(na_n^{p/2}\right)^{-1} \left(\sum_{j=1}^{n} a_j^{-2}\right)^2 \to 0 \quad as \quad n \to \infty,$$

then

$$a_n^{p/2} \sigma_n^{-1}(f_0)\left(U_n - \Theta(f_0)\right) \xrightarrow{d} N(0,1),$$

$$\gamma_s(n) = \frac{1}{n} \sum_{i=1}^{n} a_{ni}^{ps}, \quad s = 1,2, \quad \Theta(f_0) = \gamma_1 \int f(x) r(x) dx \int K^2(u) du,$$

$$\sigma_n^2(f_0) = a_n^{-p} d_n^2(f_0),$$

$$d_n^2(f_0) = \frac{2}{n^2} \iint f_0^2(x) \left(\sum_{i=1}^{n} a_i^p K_0(a_i(x-y))\right)^2 r(x) r(y) dx\, dy, \quad K_0 = K * K.$$

Also,

$$2\gamma_1^2 \int f_0^2(x) r^2(x) dx \int K_0^2(u) du \le \underline{\lim}_{n\to\infty} \sigma_n^2(f_0) \le \overline{\lim}_{n\to\infty} \sigma_n^2(f_0) \le 2\gamma_1 \int f_0^2(x) r^2(x) dx \int K_0^2(u) du. \tag{1}$$

Theorem 1 allows us to construct a goodness-of-fit test of asymptotic level α for checking the hypothesis H_0 by which $f(x) = f_0(x)$. After that we calculate U_n and reject the hypothesis H_0 if

$$U_n \ge d_n(\alpha) = \gamma_1 \int f_0(x) r(x) dx \int K^2(u) du + \lambda_\alpha a_n^{-p/2} \sigma_n(f_0), \tag{2}$$

where λ_α is a quantile of level α of a standard normal distribution.

2. Now we will investigate the asymptotic property of test (2) (i.e. the behavior of the power function as $n \to \infty$).

In the first place we consider the question whether the test is consistent. The following statement is true.

Theorem 2. *Let all the conditions of Theorem* 1 *be fulfilled. Then*

$$\Pi_n(f_1) = P_{H_1}\{U_n \ge d_n(\alpha)\} \to 1$$

as $n \to \infty$, *i.e. the test defined in* (2) *is consistent against any alternative* $H_1:\ f_1(x) \ne f_0(x)$,

$$\Delta = \int (f_1(x) - f_0(x))^2 r(x) dx > 0.$$

Proof. We have

$$\Pi_n(f_1) = P_{H_1}\left\{na_n^{-p} \int (f_n(x) - f_0(x))^2 r(x) dx \ge \Theta(f_0) + \sigma_n(f_0) a_n^{-p/2} \lambda_\alpha\right\} =$$

$$= P_{H_1}\left\{na_n^{-p} \int (f_n(x) - f_1(x))^2 r(x) dx \ge \Theta(f_0) + \sigma_n(f_0) a_n^{-p/2} \lambda_\alpha - na_n^{-p}\Delta - 2\int (f_n(x) - f_1(x)) \varphi(x) r(x) dx \cdot \frac{n}{a_n^p}\right\} =$$

$$= P_{H_1}\left\{ \frac{a_n^{p/2}\left(U_n^{(1)} - \Theta(f_1)\right)}{\sigma_n(f_1)} \geq \right.$$

$$\left. \geq - a_n^{p/2}\frac{\Theta(f_1)}{\sigma_n(f_1)} + \frac{a_n^{p/2}}{\sigma_n(f_1)}\left(\Theta(f_1) + \sigma_n(f_0)a_n^{-p/2}\lambda_\alpha - \frac{n}{a_n^p}\Delta - 2\int (f_n(x) - f_1(x))r(x)dx \cdot \frac{n}{a_n^p}\right)\right\} =$$

$$= P_{H_1}\left\{ a_n^{p/2}\sigma_n^{-1}(f_1)\left(U_n^{(1)} - \Theta(f_1)\right) \geq \right.$$

$$\left. \geq -\frac{n}{a_n^{p/2}}\left(\frac{1}{\sigma_n(f_1)}\Delta + \left(\Theta(f_1) - \Theta(f_0)\right)\frac{a_n^p}{n}\frac{1}{\sigma_n(f_1)} + 2\sigma_n^{-1}(f_1)\int (f_n(x) - f_1(x))\varphi(x)dx\right)\right\},$$

where

$$U_n^{(1)} = n\, a_n^{-p}\int (f_n(x) - f_1(x))^2 r(x)\,dx,$$

$$\varphi(x) = (f_1(x) - f_0(x))r(x).$$

Now we will show that $\int (f_n(x) - f_1(x))\varphi(x)dx \xrightarrow{P} 0$. Indeed,

$$\int (f_n(x) - f_1(x))\varphi(x)dx = B_{1n} + B_{2n},$$

$$B_{1n} = \sum_{j=1}^{n}\xi_j, \quad \xi_j = \frac{a_j^p}{n}\int \left[K\left(a_j(x - X_j)\right) - E_1 K\left(a_j(x - X_j)\right)\right]\varphi(x)\,dx,$$

$$B_{2n} = \int\left(\frac{1}{n}\sum_{j=1}^{n} a_j^p E_1 K\left(a_j(x - X_j)\right) - f_1(x)\right)\varphi(x)dx,$$

where $E_1(\cdot)$ is expectation for the hypothesis H_1.

By virtue of the assumption, for $f_1(x)$ and $K(x)$ we obtain

$$B_{2n} = O\left(\frac{1}{n}\sum_{j=1}^{n} a_j^{-2}\right). \tag{3}$$

Next, it can be easily checked that

$$\sum_{j=1}^{n} D\xi_j = n^{-2}\left\{\sum_{j=1}^{n}\int f_1(t)dt\left(\int K(u)\varphi\left(t + \frac{u}{a_j}\right)\right)^2 - \sum_{j=1}^{n}\left(\int f_1(x)dx\int K(u)\varphi\left(x + \frac{u}{a_j}\right)\right)^2\right\} \sim$$

$$\sim n^{-1}\left(\int \varphi^2(x) f_1(x)dx - \left(\int \varphi(x) f_1(x)dx\right)^2\right) \tag{4}$$

and

$$\frac{\sum_{k=1}^{n} E_1\left|\xi_k - M\,\xi_k\right|^{2+\delta}}{\left(\sum_{k=1}^{n} D\,\xi_k\right)^{1+\delta/2}} \to 0.$$

Therefore

$$\frac{B_{1n}}{\sqrt{DB_{1n}}} \xrightarrow{d} N(0,1). \tag{5}$$

But

$$\int (f_n(x) - f_1(x))\varphi(x)dx = \left[\frac{B_{1n}}{\sqrt{DB_{1n}}}\right]\sqrt{DB_{1n}} + O\left(\frac{1}{n}\sum_{j=1}^{n} a_j^{-2}\right)$$

Hence (3), (4) and (5) imply that

$$\Pi_n(f_1) = P_{H_1}\left\{a_n^{1/2}\left(\frac{U_n^{(1)} - \Theta(f_1)}{\sigma_n(f_1)}\right) \geq -\frac{n}{a_n^{p/2}}\left(\frac{1}{\sigma_n(f_1)}\Delta + \frac{1}{\sigma_n(f_1)}\left(\frac{B_{1n}}{\sqrt{DB_{1n}}}\right)\sqrt{DB_{1n}} + \right.\right.$$

$$\left.\left. + O\left(\frac{1}{n}\sum_{j=1}^{n} a_j^{-2}\right) + (\Theta(f_0) - \Theta(f_1))\frac{a_n^p}{n} + \frac{\sigma_n(f_0)}{\sigma_n(f_1)}\frac{a_n^{p/2}}{n}\lambda_\alpha\right)\right\} =$$

$$= P_{H_1}\left\{a_n^{p/2}\frac{U_n^{(1)} - \Theta(f_1)}{\sigma_n(f_1)} \geq -\frac{n}{a_n^{p/2}}\left(\frac{1}{\sigma_n(f_1)}\Delta + o_p(1)\right)\right\}.$$

Since $a_n^{p/2}\sigma_n^{-1}(f_1)\left(U_n^{(1)} - \Theta(f_1)\right)$ has an asymptotically normal distribution $(0,1)$ for the hypothesis H_1, $na_n^{-p/2} \to \infty$, $\frac{1}{n}\sum_{j=1}^{n} a_j^{-2} \to 0$, $DB_{1n} = O\left(\frac{1}{n}\right)$ and $0 < c_0 \leq \sigma_n(f_j) \leq c_1 < \infty$, $j = 0,1$, we have $\Pi_n(f_1) \to 1$ as $n \to \infty$.

Thus for any fixed alternative the power of a test based on U_n tends to 1 as $n \to \infty$. However, if with a change of n the alternative changes so that it tends to the basic hypothesis H_0, then the power of the goodness-of-fit test will not necessarily tend to 1. This certainly depends on the convergence of the tendency of the alternative to the zero hypothesis.

Let us assume now that the hypothesis H_0 we are checking is not true; actually, we have the hypothesis

$$H_n: \quad f_{1n}(x) = f_0(x) + \gamma_n\varphi(x) + o(\gamma_n), \quad \gamma_n \downarrow 0, \quad x \in R_1 = (-\infty, \infty), \quad \int \varphi(x)dx = 0.$$

Theorem 3. *Let $K(x)$ and $f_{1n}(x)$ satisfy the conditions of Theorem 1. If $a_n = n^\delta$ and $\gamma_n = n^{-1/2+\delta/4}$, $\frac{2}{9} < \delta < \frac{1}{2}$, then*

$$P_{H_n}\{U_n \geq d_n(\alpha)\} \sim 1 - \Phi\left(\lambda_\alpha - \frac{1}{\sigma_n(f_0)}\int \varphi^2(u)r(u)du\right),$$

$$\Phi(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u} \exp\left\{-\frac{x^2}{2}\right\}dx$$

as $n \to \infty$.

Proof. We have

$$P_{H_n}\{U_n \geq d_n(\alpha)\} =$$

$$= P_{H_n}\left\{\frac{a^{1/2}\left(U_n^{(1)} - \Theta(f_{1n})\right)}{\sigma_n(f_{1n})}\frac{\sigma_n(f_{1n})}{\sigma_n(f_0)} \geq \lambda_\alpha + (\Theta(f_0) - \Theta(f_{1n}))\frac{a_n^{1/2}}{\sigma_n(f_0)} - a_n^{1/2}\frac{A_{n1}}{\sigma_n(f_0)} + \frac{a_n^{1/2}A_{n2}}{\sigma_n(f_0)}\right\},$$

where

$$A_{n1} = \frac{n}{a_n}\int (f_{1n}(x) - f_0(x))^2 r(x)dx,$$

$$A_{n2} = 2\frac{n}{a_n}\int (f_n(x) - f_{1n}(x))(f_{1n}(x) - f_0(x))r(x)dx,$$

$$U_n^{(1)} = \frac{n}{a_n}\int (f_n(x) - f_{1n}(x))^2 r(x)dx.$$

It is easy to check

$$\frac{\sigma_n^2(f_{1n})}{\sigma_n^2(f_0)} \to 1 \quad \text{as} \quad n \to \infty . \tag{6}$$

By virtue of Theorem 1, for $\frac{2}{9} < \delta < \frac{1}{2}$ it can be easily checked that

$$\frac{U_n^{(1)} - \Theta(f_{1n})}{\sigma_n(f_{1n})} \xrightarrow{d} N(0,1). \tag{7}$$

Further, since $\frac{2}{9} < \delta < \frac{1}{2}$, from (1) we obtain

$$a_n^{1/2}\left(\Theta(f_0) - \Theta(f_{1n})\right)\sigma_n^{-1}(f_0) \to 0 \tag{8}$$

and, moreover,

$$\frac{a_n^{1/2}}{\sigma_n(f_0)} A_{n1} = \frac{n}{\sigma_n(f_0) a_n^{1/2}} \gamma_n^2 \int \varphi^2(x) r(x) dx + o(1) = \frac{1}{\sigma_n(f_0)} \int \varphi^2(u) r(u) du + o(1). \tag{9}$$

Denote

$$B_{n1} = \frac{n}{a_n} \int \left| (f_n(x) - E_1 f_n(x))(f_{1n}(x) - f_0(x)) r(x) \right| dx,$$

$$B_{n2} = \frac{n}{a_n} \int \left| E_1 f_n(x) - f_{1n}(x) \right| \cdot \left| f_{1n}(x) - f_0(x) \right| r(x) dx.$$

It is not difficult to establish that

$$E_1 f_n(x) - f_{1n}(x) = O\left(\frac{1}{n} \sum_{j=1}^{n} a_j^{-2} \right).$$

Hence it follows that

$$\frac{a_n^{1/2}}{\sigma_n(f_0)} B_{n2} \le c \frac{n}{\sigma_n(f_0) a_n^{1/2}} O\left(\frac{1}{n} \sum_{k=1}^{n} a_k^{-2} \right) \gamma_n = O\left(n^{\frac{2-9\delta}{4}} \right), \quad \delta > \frac{2}{9}. \tag{10}$$

Now let us consider B_{n1}. We have

$$\frac{a_n^{1/2}}{\sigma_n(f_0)} E|B_{n1}| \le 2 \frac{n}{\sigma_n(f_0) a_n^{1/2}} E \int \left| f_n(x) - E_1 f_n(x) \right| \cdot \left| f_{1n}(x) - f_0(x) \right| r(x) dx \le$$

$$\le 2 \frac{1}{\sigma_n(f_0)} \frac{n}{a_n^{1/2}} \left\{ \frac{1}{n^2} \sum_{k=1}^{n} a_k^2 \int f_{1n}(y) dy \left(\int K(a_k(x-y)) (f_{1n}(x) - f_0(x)) r(x) dx \right)^2 \right\}^{1/2} \le$$

$$\le c_2 \frac{1}{\sigma_n(f_0)} \frac{n}{a_n^{1/2}} \left\{ \frac{1}{n} \gamma_n^2 \right\}^{1/2} \le c_3 \sqrt{n}\, a_n^{-1/2} \gamma_n = c_4 n^{-\delta/4} \to 0 . \tag{11}$$

From (10) and (11) it follows that

$$\frac{a_n^{1/2}}{\sigma_n(f_0)} A_{n2} = O\left(n^{\frac{2-9\delta}{4}} \right) + O\left(n^{-\delta/4} \right). \tag{12}$$

By combining relations (7), (8), (9) and (12) we finally obtain

$$P_{H_n}\{U_n \ge d_n(\alpha)\} \sim 1-\Phi\left(\lambda_\alpha - \frac{1}{\sigma_n(f_0)}\int \varphi^2(u)r(u)\,du\right)$$

as $n \to \infty$.

Remark 1. If the smoothness parameter $a_k \equiv a_n$, $k = 1,2,\ldots$, then by Theorem 2 we obtain an analogous limit power function for the alternative H_n of the Rosenblatt-Bickel goodness-of-fit test [4]:

$$P_{H_n}\{T_n \ge d_n^{(1)}(\alpha)\} \sim 1-\Phi\left(\lambda_\alpha - \frac{1}{\sigma_0}\int \varphi^2(x)r(x)\,dx\right),$$

$$T_n = \frac{n}{a_n}\int (f_n(x) - f_0(x))^2\, r(x)\,dx,$$

where $f_n(x)$ is an estimate of the Rosenblatt-Parzen density,

$$d_n^{(1)}(\alpha) = \int f_0(x)r(x)\,dx \int K^2(u)\,du + a_n^{-1/2}\lambda_\alpha\,\sigma_0,$$

$$\sigma_0^2 = 2\int f_0^2(x)r^2(x)\,dx \int K_0^2(u)\,du.$$

If $\lim\limits_{n\to\infty}\sigma_n^2(f_0) = \sigma_1^2$, then inequality (1) implies that $\gamma_1^2\sigma_0^2 \le \sigma_1^2 \le \gamma_1\sigma_0^2$, $0 < \gamma_1 \le 1$. Therefore, the test based on U_n for the alternatives H_n and $\gamma_1 \ne 1$, goodness-of-fit tests based on T_n are more powerful and, moreover, they are asymptotically equivalent for $\gamma_1 = 1$.

3. Let us now introduce into the consideration the alternatives we call “singular” ([5], [6]):

$$H_n : f_{1n}(x) = f_0(x) + \alpha_n \varphi\left(\frac{x-l}{\gamma_n}\right) + o(\alpha_n \cdot \gamma_n),$$

where $\alpha_n \downarrow 0$, $\gamma_n \downarrow 0$, the function $\varphi(x)$ is bounded and has bounded derivatives up to second order, $\varphi^{(2)}(x) \in L_1(-\infty,\infty)$ and $\int \varphi(x)dx = 0$, l is some continuity point of $r(x)$ such that $r(l) \ne 0$ (see also [8]).

Theorem 4. *Let* $K(x)$ *and* $f_{1n}(x)$ *satisfy the conditions of Theorem* 1. *If*

$$\alpha_n \cdot \gamma_n = o(n^{-1/2}), \quad n a_n^{-1/2}\alpha_n^2\gamma_n \to c_0 \ne 0, \quad \frac{1}{\alpha_n}\frac{1}{n}\sum_{k=1}^{n} a_k^{-2} \to 0,$$

and

$$\gamma_n^{-2}\left(\frac{1}{n}\sum_{k=1}^{n} a_k^{-4}\right)^{1/2} \to 0,$$

then

$$P_{H_n}\{U_n \ge d_n(\alpha)\} \sim 1-\Phi\left(\lambda_\alpha - \frac{c_0 r(l)}{\sigma_n(f_0)}\int \varphi^2(u)\,du\right)$$

as $n \to \infty$.

Proof. We write U_n as a sum

$$U_n = U_n^{(1)} + A_{n1} + A_{n2},$$

where

$$U_n^{(1)} = \frac{n}{a_n}\int (f_n(x) - f_{1n}(x))^2\, r(x)\,dx,$$

$$A_{n1} = \frac{n}{a_n} \int \left(f_{1n}(x) - f_0(x)\right)^2 r(x)\,dx,$$

$$A_{n2} = 2\frac{n}{a_n} \int \left(f_n(x) - f_{1n}(x)\right)\left(f_{1n}(x) - f_0(x)\right) r(x)\,dx.$$

Therefore

$$P_{H_n}\left\{U_n \geq d_n(\alpha)\right\} =$$

$$= P_{H_1}\left\{\frac{a_n^{1/2}\left(U_n^{(1)} - \Theta(f_{1n})\right)}{\sigma_n(f_{1n})}\frac{\sigma_n(f_{1n})}{\sigma_n(f_0)} \geq \lambda_\alpha + \left(\Theta(f_0) - \Theta(f_{1n})\right)\frac{a_n^{1/2}}{\sigma_n(f_0)} - a_n^{1/2}\frac{A_{n1}}{\sigma_n(f_0)} + \frac{a_n^{1/2}A_{n2}}{\sigma_n(f_0)}\right\}.$$

Analogously to (6) we have

$$\frac{\sigma_n(f_n)}{\sigma_n(f_0)} \to 1. \tag{13}$$

Further, by virtue of Theorem 1, we have

$$P_{H_n}\left\{\frac{a_n^{1/2}\left(U_n^{(1)} - \Theta(f_{1n})\right)}{\sigma_n(f_{1n})} < x\right\} \to \Phi(x) \tag{14}$$

as $n \to \infty$.

Let us now show that $\frac{a_n^{1/2}A_{n2}}{\sigma_n(f_0)} \to 0$. We have

$$a_n^{1/2}E\left|A_{n2}\right| \leq 2\,n\,a_n^{-1/2}E\left|\int \left(f_n(x) - E\,f_{1n}(x)\right)\left(f_{1n}(x) - f_0(x)\right) r(x)\,dx\right| +$$
$$+ 2\,n\,a_n^{-1/2}\int \left|E\,f_n(x) - f_{1n}(x)\right| \cdot \left|f_{1n}(x) - f_0(x)\right|\, r(x)\,dx,$$

and also

$$na_n^{-1/2}\int \left|E\;f_n(x) - f_{1n}(x)\right| \cdot \left|f_{1n}(x) - f_0(x)\right|\, r(x)\;dx = O\left(\frac{1}{\alpha_n}\frac{1}{n}\sum_{k=1}^{n} a_k^{-2}\right).$$

Further,

$$n\,a_n^{-1/2}E\left|\int \left(f_n(x) - E\,f_n(x)\right)\left(f_{1n}(x) - f_0(x)\right) r(x)\,dx\right| \leq$$

$$\leq n\,a_n^{-1/2}\left\{\frac{1}{n^2}\sum_{k=1}^{n} a_k^2 \int f(u)\,du \left[\int K\left(a_k(x-u)\right)\left(f_{1n}(x) - f_0(x)\right) r(x)\,dx\right]^2\right\}^{1/2} \leq$$

$$\leq c_5 n\,\alpha_n\,a_n^{-1/2}\left\{\frac{1}{n}\int f(u)\varphi^2\left(\frac{u-l}{\gamma_n}\right)du + \frac{1}{\gamma_n^4 n^2}\sum_{k=1}^{n} a_k^{-4}\int f(u)\,du\left[\int\int_0^1 t^2 K(t)\varphi^{(2)}\left(\frac{u-l}{\gamma_n} + \frac{zt}{a_k\gamma_n}\right)dt\,dz\right]^2\right\}^{1/2}.$$

Hence, using the generalized Minkowski's inequality [7], we obtain

$$n\;a_n^{-1/2}E\left|\int \left(f_n(x) - E\,f_n(x)\right)\left(f_{1n}(x) - f_0(x)\right) r(x)\,dx\right| \leq c_6\,a_n^{-1/4} + c_7\,\gamma_n^{-2}\left(\frac{1}{n}\sum_{k=1}^{n} a_k^{-4}\right)^{1/2}.$$

Thus

$$a_n^{1/2}E\left|A_{n2}\right| = O\left(a_n^{-1/4}\right) + O\left(\gamma_n^{-2}\left(\frac{1}{n}\sum_{k=1}^{n} a_k^{-4}\right)^{1/2}\right) + O\left(\alpha_n^{-1}\frac{1}{n}\sum_{k=1}^{n} a_k^{-2}\right) \tag{15}$$

From the condition $n\,\alpha_n^2\,a_n^{-1/2}\,\gamma_n \to c_0 \neq 0$ it is not difficult to establish that

$$a_n^{1/2} A_{1n} = n a_n^{-1/2} \int (f_{1n}(x) - f_0(x))^2 r(x) dx = n a_n^{-1/2} \alpha_n^2 \gamma_n \cdot \frac{1}{\gamma_n} \int \varphi^2 \left(\frac{x-l}{\gamma_n} \right) r(x) dx \to$$

$$\to c_0 r(l) \int \varphi^2(u) du, \tag{16}$$

and likewise it is easy to see that

$$(\Theta(f_0) - \Theta(f_{1n})) \frac{a_n^{1/2}}{\sigma_n(f_0)} = o\left(\sqrt{\frac{a_n}{n}} \right). \tag{17}$$

Finally, (13), (14), (15), (16) and (17) imply

$$P_{H_n} \{U_n \geq d_n(\alpha)\} \sim 1 - \Phi\left(\lambda_\alpha - \frac{c_0\, r(l)}{\sigma_n(f_0)} \int \varphi^2(u) du \right).$$

The conditions of Theorem 4 for a_n, α_n, γ_n are fulfilled if, say, we assume that $a_n = n^\delta$, $\alpha_n = n^{-\alpha}$, $\gamma_n = n^{-\beta}$ for $\frac{\delta}{2} = 1 - 2\alpha - \beta$, $\alpha + \beta > \frac{1}{2}$, $0 < \delta \leq \frac{1}{2}$, $\beta < \delta$, $\delta > \frac{\alpha}{2}$, while the conditions imposed on α, β and δ are fulfilled, for example, for

$$\delta = \frac{1}{2}, \quad \beta = \frac{5}{12}, \quad \alpha = \frac{1}{6},$$
$$\delta = \frac{1}{4}, \quad \beta = \frac{1}{5}, \quad \alpha = \frac{27}{80},$$
$$\delta = \frac{1}{5}, \quad \beta = \frac{1}{6}, \quad \alpha = \frac{11}{30}$$

and so on.

Remark 2. If the smoothness parameter $a_k \equiv a_n$, $k = 1,2,\ldots$, then by Theorem 4 we obtain the limit power for alternative H_n of the goodness-of-fit test T_n [8]

$$P_{H_n} \left\{ T_n \geq d_n^{(1)}(\alpha) \right\} \sim 1 - \Phi\left(\lambda_\alpha - \frac{c_0\, r(l)}{\sigma_n(\sigma_0)} \int \varphi^2(u) du \right).$$

If $\lim\limits_{n\to\infty} \sigma_n^2 = \sigma_1^2$, then from inequality (1) we have $\gamma_1^2 \sigma_0^2 \leq \sigma_1^2 \leq \gamma_1 \sigma_0^2$, $0 < \gamma_1 \leq 1$. Therefore, for $\gamma_1 \neq 1$, goodness-of-fit tests based on U_n are more powerful than those based on T_n. Moreover, integrating $f_{1n}(x)$, we establish that the alternatives differ from the zero hypothesis by a value of order $\alpha_n \gamma_n = o\left(\frac{1}{\sqrt{n}} \right)$. Therefore goodness-of-fit tests based on a difference between empirical distribution functions like for example, tests ω_n^2 and Kolmogorov-Smirnov type tests cannot differentiate between “singular” hypotheses and the basic one. Hence, by virtue of Theorem 4, for “singular” alternatives the tests based on U_n are more powerful than those of the type mentioned above.

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ე. ნადარაია*, პ. ბაბილუა**

* *აკადემიის წევრი, ი. ჯავახიშვილის სახელობის თბილისის სახელმწიფო უნივერსიტეტი*
** *ი. ჯავახიშვილის სახელობის თბილისის სახელმწიფო უნივერსიტეტი*

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