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Mathematics

On Solving the Internal Three-Dimensional Dirichlet Problem for a Harmonic Function by the Method of Probabilistic Solution

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ABSTRACT. The algorithm of solving the internal three-dimensional Dirichlet boundary value problem for a harmonic function by the method of probabilistic solution is given. The algorithm is based on a computer simulation of the Wiener process. To illustrate the effectiveness and simplicity of the proposed method a numerical example is considered.

Key words: *Three-dimensional Dirichlet problem, harmonic function, probabilistic solution, computer simulation.*

Let D be a finite domain in the Euclidian space E_3, bounded by one closed piecewise smooth surface S (i.e., $S = \bigcup_{j=1}^{m} S^j$) is given, where each part S^j is a smooth surface. Besides, we assume: 1) equations of the parts S^j are given; 2) edges of the surface S are piecewise smooth contours; 3) for the surface S it is easy to show that a point $x = (x_1, x_2, x_3) \in E_3$ lies in $\overline{D}$ or not. For the Laplace equation we consider the Dirichlet boundary value problem.

Problem A. Find a function $u(x) \equiv u(x_1, x_2, x_3) \in C^2(D) \cap C(\overline{D})$ satisfying the conditions:

$$\Delta u(x) = 0, \qquad x \in D,$$
$$u(y) = g(y), \qquad y \in S,$$

where $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and $g(y) \equiv g(y_1, y_2, y_3)$ is a continuous real function on S.

It is known [1,2] that Problem A is correct, i.e., its solution exists, is unique and depends on data continuously. It should be noted that the labouriousness of solving problems sharply increases along with the dimension of the problems considered. Therefore, as a rule, one fails to develop standard methods for solving a wide class of multidimensional problems with the same high accuracy as in the one-dimensional case. In the example the exact solution of Problem A for a disk is written by one-dimensional Poisson's integral and in the case of sphere by two-dimensional Poisson's integral [2-4]:

$$u(x) = \frac{1}{4\pi r} \iint_{|y|=r} \frac{r^2 - |x|^2}{|x-y|^3} g(y) dys, \qquad |x| < r, \tag{1}$$

where r is the radius of the sphere with the center at the origin O $(r = |y| = |Oy|,\ y \in S)$. Integral (1) loses sense when $|x| = r$. However, it is proved [4] that $u(x) \to g(y)$ when $x \to y$ $(x \in D)$. A simple example, given by us, shows the difficulty in determining the solution with high accuracy of the Dirichlet problem when the dimension increases.

In monograph [5] it is proved that in the case of the finite domain D the solution of the Problem A at the fixed point $x \in D$ has the form

$$u(x) = M_x g(x(t)), \tag{2}$$

where $M_x g(x(t))$ is the expectation value of the boundary function $g(y)$ at the random intersection points of the Wiener process and the boundary S; t is the moment of first exit of the Wiener process $x(t) = (x_1(t), x_2(t), x_3(t))$ from the domain D. It is assumed that the starting point of the Wiener process is always $x(t_0) = (x_1(t_0), x_2(t_0), x_3(t_0)) \in D$, where the value of the desired function is being determined. If the number N of the random points $y^i = (y_1^i, y_2^i, y_3^i) \in S$ $(i = 1, 2, ..., N)$ is sufficiently large, then according to the law of large numbers, from (2) we have

$$u(x) \approx u_N(x) = \frac{1}{N}\sum_{i=1}^{N} g(y^i) \tag{3}$$

or $u(x) = \lim u_N(x)$ for $N \to \infty$, in the probabilistic sense. Thus, in the presence of the Wiener process the approximate value of the probabilistic solution to the Problem A at a point $x \in D$ we calculate by formula (3).

Analogously to the plane case (see [6,7]), probabilistic solving of the Problem A consists in realization of the Wiener process using the three-dimensional generator, which gives three independent values $w_1(t), w_2(t), w_3(t)$. In the considered case the Wiener process is realized by computer simulation. In particular, for the computer simulation of the Wiener process we use the following recursion relations:

$$\begin{aligned} x_1(t_k) &= x_1(t_{k-1}) + w_1(t_k)/kv, \\ x_2(t_k) &= x_2(t_{k-1}) + w_2(t_k)/kv, \\ x_3(t_k) &= x_3(t_{k-1}) + w_3(t_k)/kv, \\ &(k = 1, 2, ...), \end{aligned} \tag{4}$$

with the help of which coordinates of a current point $(x_1(t_k), x_1(t_k), x_1(t_k))$ are being determined. In (4) $w_1(t_k), w_2(t_k), w_3(t_k)$ are three normally distributed independent random numbers for k-th step, with zero means and variances one; kv is a number of the quantification and when $kv \to \infty$, then the discrete Wiener process approaches the continuous Wiener process. In the computer the random process is simulated at each step of the walk and continues until it crosses the boundary. In our case noted random numbers are generated in the environment of the MATLAB system.

Example. The domain D is the interior of the three-dimensional ellipsoid S:

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1,$$

where values a, b, c are semi-axes of the ellipsoid, and $x(x_1, x_2, x_3)$ is a current point of the surface S.

To define the intersection points $y^i = (y_1^i, y_2^i, y_3^i)$ $(i=1,2,...,N)$ of the Wiener process and of the surface S, we operate in the following way. During the realization of the Wiener process, for each current point $x(t_k)$, defined from (3), its location with respect to the boundary S is checked, i.e., for the point $x(t_k)$ the value

$$d=\frac{x_1^2(t_k)}{a^2}+\frac{x_2^2(t_k)}{b^2}+\frac{x_3^2(t_k)}{c^2}$$

is calculated and the conditions: $d=1$, $d<1$ or $d>1$ are checked. If $d=1$, then $x(t_k)\in S$ and $y^i=x(t_k)$. If $d<1$ then $x(t_k)\in D$, and if $d>1$, then $x(t_k)\notin \overline{D}$. Let $x(t_{k-1})\in D$ for the moment $t=t_k$. In this case, for an approximate definition of the point y^i, a parametric equation of a line l passing through the points $x(t_{k-1})$ and $x(t_k)$ is written in the first place. After this the intersection points x^* and x^{**} of the line l of the surface S are defined. In the role of the point y^i from the points x^* and x^{**} a point is taken for which $|x(t_k)-x|$ is minimal.

The results of numerical experiments for a triaxial ellipsoid with semi-axes: $a=3,\ b=2,\ c=1$ are given in Table 1.

As boundary function $g(\tau)$ is taken $g(\tau)=\dfrac{1}{|y-y^0|}$, $|y-y^0|=\sqrt{(y_1-y_1^0)^2+(y_2-y_2^0)^2+(y_3-y_3^0)^2}$, where $y^0=(0,0,10)$. It is evident that in the considered case the exact solution to Problem A is $u(x)=\dfrac{1}{|x-y^0|}$, $x\in D$.

In Table 1: N is the Wiener process realization number for the given points $x^j=(x_1^j,x_1^j,x_1^j)\in D,\quad (j=1,2,3)$; $\Delta^j=|u_1(x^j)-u(x^j)|$, where $u_1(x^j)$ is the approximate solution to Problem A at the point x^j, which is defined by formula (3).

Table 1

	(0, 0, 0)		(0.5, 0.5, 0.5)		(0, 0, 0.999)	
	(kv=100)	(kv=200)	(kv=100)	(kv=200)	(kv=100)	(kv=200)
N	Δ^1	Δ^1	Δ^2	Δ^2	Δ^3	Δ^3
500	1.51E-04	7.96E-04	5.46E-04	3.18E-04	1.32E-04	3.77E-05
1000	5.67E-04	2.49E-04	1.84E-04	8.77E-05	5.54E-05	5.98E-05
1500	2.88E-04	2.79E-05	1.64E-04	3.64E-05	8.70E-05	4.37E-05
2000	1.46E-04	6.81E-05	4.99E-05	3.65E-05	4.88E-05	1.90E-05
3000	1.09E-04	7.15E-06	6.68E-05	8.19E-05	8.22E-05	4.79E-05
4000	8.57E-05	6.60E-05	7.61E-06	4.08E-05	6.67E-05	5.56E-05
5000	5.74E-05	4.87E-05	7.50E-05	5.14E-05	8.80E-05	3.72E-05
10000	3.17E-05	6.02E-05	1.01E-04	2.21E-05	7.28E-05	3.57E-05
20000	1.72E-06	6.01E-05	2.15E-05	2.38E-05	8.17E-05	3.05E-05
30000	4.53E-06	3.94E-05	3.69E-06	1.02E-05	7.52E-05	3.48E-05
40000	2.42E-05	2.36E-05	3.04E-05	3.07E-06	7.73E-05	4.17E-05
50000	2.32E-05	1.13E-05	6.44E-05	3.08E-05	8.37E-05	4.23E-05
75000	4.54E-05	5.80E-06	3.00E-05	2.93E-05	8.54E-05	4.11E-05
100000	2.42E-05	1.92E-05	6.35E-05	2.43E-05	8.69E-05	3.76E-05
150000	3.49E-05	3.57E-05	2.13E-05	6.27E-06	8.02E-05	3.96E-05
200000	2.48E-05	2.89E-05	1.64E-05	6.13E-06	8.75E-05	4.05E-05

From Table 1 it is seen that $\Delta^j\to 0$, when $N\to\infty$, in the probabilistic sense.

მათემატიკა

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