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Mathematics

On Approximate Solution of an Inverse Problem for Linear Delay Differential Equations

Akaki Arsenashvili*, **Abdeljalil Nachaoui****, **Tamaz Tadumadze***

* *I. Javakhishvili Tbilisi State University, Tbilisi, Georgia*
** *University of Nantes, Nantes, France*

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ABSTRACT. The inverse problem for linear control delay differential equations with non-fixed initial function, non-fixed initial moment and vector is posed. With a view to an approximate solution of the inverse problem, its corresponding "regularization" optimal control problem is considered and appropriate necessary conditions of optimality are formulated. The inverse problem, when the initial moment and vector are fixed, is solved approximately by iteration method. *© 2008 Bull. Georg. Natl. Acad. Sci.*

Key words: *inverse problem, linear delay equations, optimality conditions, approximation, iteration method.*

1. Statement of the inverse problem and optimality conditions for the corresponding regularization optimal problem

Let $s_0 < s_1 < s_2 < t_1$ and $\tau > 0$ be given numbers, with this $t_1 - s_1 > \tau$; R_x^n be an $n-$dimensional vector space of points $x = (x^1,...,x^n)^T$, $\|x\|^2 = \sum_{i=1}^{n}(x^i)^2$; $M \subset R_x^n$, $X \subset R_x^n$, $U \subset R_u^r$ be convex compact sets; Δ be the set of measurable initial functions $\varphi : [s_0 - \tau, s_1] \to M$; Ω be the set of measurable control functions $u : I = [s_0, t_1] \to U$.

To each element $w = (t_0, x_0, \varphi(\cdot), u(\cdot)) \in W = [s_0, s_1] \times X \times \Delta \times \Omega$ we assign the linear delay differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-\tau) + C(t)u(t) + f(t),\ t \in [t_0, t_1] \tag{1}$$

with the discontinuous initial condition

$$x(t) = \varphi(t),\ t \in [t_0 - \tau, t_0), \qquad x(t_0) = x_0, \tag{2}$$

where $A(t), B(t), C(t),\ t \in I$ are given continuous matrix functions with appropriate dimensions, $f(t) \in R_x^n,\ t \in I$ is a given continuous function.

The initial condition (2) is called discontinuous because, in general, $x(t_0) \neq \varphi(t_0)$.

Let $w = (t_0, x_0, \varphi(\cdot), u(\cdot)) \in W$. A function $x(t) = x(t; w) \in R_x^n,\ t \in [t_0 - \tau, t_1]$ is called a solution corresponding to the element w, if it satisfies conditions (2) on the interval $[t_0 - \tau, t_0]$, is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) everywhere.

It is clear that the solution $x(t), t \in [t_0 - \tau, t_1]$, in general, is discontinuous at the point t_0 .

It is known [1] that for every element $w=(t_0,x_0,\varphi(\cdot),u(\cdot))\in W$ there exists unique solution $x(t;w)$ defined on the interval $[t_0-\tau,t_1]$.

Introduce the set

$$Z=\{z(t),t\in[s_2,t_1]:\exists\, w\in W,\, x(t;w)=z(t),t\in[s_2,t_1]\}.$$

The inverse problem: Let $z(\cdot)\in Z$ be a given function. Find element $w\in W$ such that the function $x(t;w)$ satisfies the condition

$$x(t;w)=z(t),t\in[s_2,t_1].$$

With a view to approximate solution of the inverse problem we consider the following regularization optimal control problem with non-fixed initial moment and with the discontinuous initial condition:

$$\dot x(t)=A(t)x(t)+B(t)x(t-\tau)+C(t)u(t)+f(t),\ t\in[t_0,t_1],\ u(\cdot)\in\Omega\ , \tag{3}$$

$$x(t)=\varphi(t),t\in[t_0-\tau,t_0),x(t_0)=x_0,\varphi(\cdot)\in\Delta,x_0\in X, \tag{4}$$

$$J(w;\delta)=\frac{1}{2}\int_{s_2}^{t_1}\|x(t;w)-z(t)\|^2dt+\delta_1\int_{t_0-\tau}^{t_0}\|\varphi(t)\|^2dt+\delta_2\int_{t_0}^{t_1}\|u(t)\|^2dt\to\min, \tag{5}$$

where $\delta_i>0,\ i=1,2$ are fixed numbers.

The problem (3)-(5) is called optimal control problem corresponding to the inverse problem.

The following theorems are valid (see [2,3]):

Theorem 1. *For the problem (3)-(5) there exists optimal element* $w_{op}^{\delta}=(t_{op}^{\delta},x_{op}^{\delta},\varphi_{op}^{\delta}(\cdot),\ u_{op}^{\delta}(\cdot))\in W$ and

$$\lim_{\delta\to0}J(w_{op}^{\delta};\delta)=0,\ \text{where}\ \delta=(\delta_1,\delta_2).$$

For simplicity we denote an optimal element by w_{op}.

Theorem 2. *Let w_{op} be an optimal element, $x_{op}(t)=x(t;w_{op})$ and $t_{op}\in(s_0,s_1)$. Moreover, the functions $u_{op}(t)$, $\varphi_{op}(t)$ and $\varphi_{op}(t-\tau)$ are continuous at the point t_{op}. Then the following conditions hold :*

1) the condition for the initial moment t_{op}

$$-\delta_1\|\varphi_{op}(t_{op}-\tau)\|^2-\delta_2\|u_{op}(t_{op})\|^2+\psi(t_{op})[A(t_{op})x_{op}(t_{op})+B(t_{op})\varphi_{op}(t_{op}-\tau)+C(t_{op})u_{op}(t_{op})+$$
$$f(t_{op})]+\psi(t_{op}+\tau)B(t_{op}+\tau)\ [x_{op}-\varphi_{op}(t_{op})]=0;$$

2) the condition for the initial vector x_{op}

$$\psi(t_{op})x_{op}=\max_{x_0\in X}\psi(t_{op})x_0;$$

3) the maximum principle for the initial function $\varphi_{op}(t)$

$$\psi(t+\tau)B(t+\tau)\varphi_{op}(t)-\delta_1\|\varphi_{op}(t)\|^2=\max_{\varphi\in M}[\psi(t+\tau)B(t+\tau)\varphi-\delta_1\|\varphi\|^2],t\in[t_{op}-\tau,t_{0p}];$$

4) the maximum principle for the control $u_{op}(t)$

$$\psi(t)C(t)u_{op}^{\delta}(t)-\delta_2\|u_{op}(t)\|^2=\max_{u\in U}[\psi(t)C(t)u-\delta_2\|u\|^2],t\in[t_{op},t_1].$$

Here $\psi(t)$ is the solution of the equation

$$\dot\psi(t)=-\psi(t)A(t)-\psi(t+\tau)B(t+\tau)+\chi(t)[x_{op}(t)-z(t)]^T,\ t\in[t_{op},t_1], \tag{6}$$

$$\psi(t)=0,\quad t\in[t_1,t_1+\tau],$$

where $\chi(t)$ is the characteristic function of the interval $[s_2,t_1]$, it is supposed that $\chi(t)[x(t)-z(t)]^T=0$, for $t\notin[s_2,t_1]$.

Some comments: In the condition 1) the expression $\psi(t_{op}+\tau)B(t_{op}+\tau)\,[x_{op}-\varphi_{op}(t_{op})]$ is the effect of discontinuity; if $u_{op}(t_{op}-)$, $\varphi_{op}(t_{op}-)$ and $\varphi_{op}(t_{op}-\tau-)$ or $u_{op}(t_{op}+)$, $\varphi_{op}(t_{op}+)$ and $\varphi_{op}(t_{op}-\tau+)$ are finite, then at the moment t_{op} instead of 1) (see Theorem 2) we have, respectively:

$$-\delta_1\|\varphi_{op}(t_{op}-\tau-)\|^2-\delta_2\|u_{op}(t_{op}-)\|^2+\psi(t_{op})[A(t_{op})x_{op}(t_{op})+B(t_{op})\varphi_{op}(t_{op}-\tau-)+$$

$$C(t_{op})u_{op}(t_{op}-)+f(t_{op})]+\psi(t_{op}+\tau)B(t_{op}+\tau)\,[x_{op}-\varphi_{op}(t_{op}-)]\le 0, \tag{7}$$

or

$$-\delta_1\|\varphi_{op}(t_{op}-\tau-)\|^2-\delta_2\|u_{op}(t_{op}-)\|^2+\psi(t_{op})[A(t_{op})x_{op}(t_{op})+B(t_{op})\varphi_{op}(t_{op}-\tau+)+$$

$$+C(t_{op})u_{op}(t_{op}+)+f(t_{op})]+\psi(t_{op}+\tau)B(t_{op}+\tau)\,[x_{op}-\varphi_{op}(t_{op}+)]\ge 0; \tag{8}$$

if $t_{op}=s_0$, then we have (7), if $t_{op}=s_1$, then we have (8).

Finally, note that a class of inverse problem for the linear delay differential equations without control function $u(t)$ is considered in [4].

The function $z(t)$, as a rule, by distinct error is before given. Thus instead of the function $z(t)$ we have a continuous function $z_0(t)$ which is an approximation to the function $z(t)$ and in general does not belong to set Z. In this case it is natural to change the above posed inverse problem by the following approximate problem: Find an element $w\in W$ such that the deviation

$$\frac{1}{2}\int_{s_2}^{t_1}\|x(t;w)-z_0(t)\|^2\,dt$$

takes the minimal value.

2. Approximate solving of the inverse problem when the initial moment and vector are fixed

Let t_0 and x_0 be fixed. In this case $W=\Delta\times\Omega$ and $w=(\varphi(\cdot),u(\cdot))$.

Theorem 3. *Let in the problem (3)-(5) t_0 and x_0 be fixed, then there exists a unique optimal element $w_{op}=(\varphi_{op}(\cdot),u_{op}(\cdot))$. For optimality of the element w_{op} the conditions 3) and 4) are necessary and sufficient.*

Iteration process of solving of the problem (3)-(5): Let $\varphi_0(\cdot)\in\Delta$ and $u_0(\cdot)\in\Omega$ be a starting approximation of the initial function and the control function.

We construct the sequences $\{\psi_k(t)\},\{\varphi_k(t)\},\{u_k(t)\},\{x_k(t)\}$ by the following iteration process:

5) for given $\varphi_k(t)$ and $u_k(t)$ obtain $x_k(t)$, the solution of the delay differential equation

$$\begin{cases}\dot x_k(t)=A(t)x_k(t)+B(t)x_k(t-\tau)+C(t)u_k(t)+f(t),\\ x_k(t)=\varphi_k(t),t\in[t_0-\tau,t_0),x(t_0)=x_0;\end{cases} \tag{9}$$

6) if a stopping criterion is satisfied, stop;

7) obtain $\psi_k(t)$, the solution of the following equation

$$\begin{cases}\dot\psi_k(t)=-\psi_k(t)A(t)-\psi_k(t+\tau)B(t+\tau)+\chi(t)[x_k(t)-z(t)]^T,\\ \psi_k(t)=0,t\in[t_1,t_1+\tau],\end{cases} \tag{10}$$

(see (6));

8) put $k=k+1$ and find the next iterates $\varphi_k(t)$ and $u_k(t)$ as the solutions of equations (11) and (12) respectively :

$$\psi_{k-1}(t)B(t+\tau)\varphi_k(t)-\delta_1\|\varphi_k(t)\|^2=\max_{\varphi\in M}[\psi_{k-1}(t)B(t+\tau)\varphi-\delta_1\|\varphi\|^2],t\in[t_0-\tau,t_0]\ , \tag{11}$$

$$\psi_{k-1}(t)C(t)u_k(t)-\delta_2\|u_k(t)\|^2=\max_{u\in U}[\psi_{k-1}(t)C(t)u-\delta_2\|u\|^2,\ t\in[t_0,t_1], \tag{12}$$

(see 3), 4) conditions of the Theorem 2);

9) go to 5).

Remark 1. Let $C(t)=(c_j^i(t)),i,j=\overline{1,n}$; $B(t)=(b_j^i(t)),i=\overline{1,n},j=\overline{1,r}$; $\psi_{k-1}(t)=(\psi_{k-1}^1(t)),\dots,\ \psi_{k-1}^n(t)$ and

$$M=\left\{\varphi=(\varphi^1,\dots,\varphi^n)^T:\ a_0^i\le\varphi^i\le b_0^i,i=\overline{1,n}\right\},\ U=\left\{u=(u^1,\dots,u^r)^T:a_1^i\le u^i\le b_1^i,i=\overline{1,r}\right\} \tag{13}$$

then we have

$$\varphi_k^i(t)=\begin{cases}a_0^i,\ if\ \Psi_{0,k-1}^i(t)/2\delta_1<a_0^i,\\ \Psi_{0,k-1}^i(t)/2\delta_1,\ if\ a_0^i\le\Psi_{0,k-1}^i(t)/2\delta_1\le b_0^i,\\ b_0^i,\ if\ \Psi_{0,k-1}^i(t)/2\delta_1>b_0^i,\end{cases} \tag{14}$$

$i=\overline{1,n}$;

$$u_k^i(t)=\begin{cases}a_1^i,\ if\ \Psi_{1,k-1}^i(t)/2\delta_2<a_1^i,\\ \Psi_{1,k-1}^i(t)/2\delta_2,\ if\ a_1^i\le\Psi_{1,k-1}^i(t)/2\delta_2\le b_1^i,\\ b_1^i,\ if\ \Psi_{1,k-1}^i(t)/2\delta_2>b_1^i,\end{cases} \tag{15}$$

$i=\overline{1,r}$,

(see 3), 4) conditions of Theorem 2).

Here

$$\Psi_{0,k-1}^i(t)=\sum_{j=1}^n\psi_{k-1}^j(t)c_i^j(t),i=\overline{1,n}\ ;\ \Psi_{1,k-1}^i(t)=\sum_{j=1}^r\psi_{k-1}^j(t)b_i^j(t),i=\overline{1,r}.$$

On the basis of Theorem 3 the following theorem is proved.

Theorem 4. *The following relations are valid:*

$$\lim_{k\to\infty}\psi_k(t)=\psi(t),\ \textit{uniformly for}\ t\in[t_0,t_1];$$

$$\lim_{k\to\infty}x_k(t)=x_{op}(t),\ \textit{uniformly for}\ t\in[t_0,t_1];$$

$$\lim_{kn\to\infty}\varphi_k(t)=\varphi_{op}(t),\ \textit{weakly in the space}\ L_1[t_0-\tau,t_0];$$

$$\lim_{k\to\infty}u_k(t)=u_{op}(t),\ \textit{weakly in the space}\ L_1[t_0,t_1].$$

Remark 2. The equation (9) and (10) can be solved approximately by the method of steps. Since $x_k(t-\tau)=\varphi_k(t-\tau)$, $t\in[t_0,t_0+\tau)$, consequently on the interval $[t_0,t_0+\tau]$ we have a linear ordinary differential equation with the initial condition $x_k(t_0)=x_0$, which can be solved, for example, using the classical Runge-Kutta method [5] combined with Hermit interpolation. The approximate solution $\tilde{x}_k(t)$, constructed on the interval $[t_0,t_0+\tau]$ by an interpolation formula and $\tilde{x}_k(t_0+\tau)$, will be used as the initial function and the initial vector on the next interval $[t_0+\tau,t_0+2\tau]$. Continuing this process, we will get an approximate solution $\tilde{x}_k(t),t\in[t_0,t_1]$ for the differential equation (9). In an analogous way, equation (10) can be solved approximately from right to left. We denote the approximate solution of equation (10) by $\tilde{\psi}_k(t)$. After this, if M and U have the form (13), then we can compute $\tilde{\varphi}_{k+1}(t)$ and $\tilde{u}_{k+1}(t)$ (see (14), (15)) corresponding to solutions $\tilde{x}_k(t)$ and $\tilde{\psi}_k(t)$.

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ა. არსენაშვილი*, ა. ნაშავი, თ. თადუმაძე***

** ი. ჯავახიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტი*
*** ნანტის უნივერსიტეტი, ნანტი, საფრანგეთი*

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