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Mathematics

Metastrategic Extensions of Lexicographic Noncooperative Game in Case of Two Players

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ABSTRACT. Informational and metastrategic extensions – metagames of lexicographic finite noncooperative game are discussed in the paper. It is proved that there always exists an equilibrium situation in the first metagames. According to an example it is shown that in the metaextensions there exists such a situation which will be at the same time equilibrium and optional according to Pareto's opinion. *© 2008 Bull. Georg. Natl. Acad. Sci.*

Key words: *noncooperative lexicographic game, equalized situation, metagames, metastrategies.*

Let us discuss n player's scalar noncooperative (noncoalition) game [1]

$$\Gamma(n) =< N, \{X_i\}_{i\in N}, \{H_i\}_{i\in N} >,$$

where N={1,2, ..., n} is the set of the players, X_i is the set of pure strategies of $i \in N$ players, H_i is the scalar payoff function of the situations of $i \in N$ players: $H_i: X=\prod_{i\in N} X_i \to R^1$. The game $\Gamma(n)$ is held in the following way: each player $i \in N$ chooses $x_i \in X_i$ strategy independently, he does not know anything about the choice of the other players.

If we discuss such case of $\Gamma(n)$ game play where each player $i \in N$ has access to some information about the decisions of their partners' choice, and according to this he has an opportunity to make some kind of information about the choosing of $x_i \in X_i$ strategy, then every such kind of game is called informational extension of $\Gamma(n)$ game.

N.Howard's metagames and metaextensions [2] represent a certain class of informational extension of noncooperative games. Informational extension of $\Gamma(n)$ game is a simple example of metaextension of $\Gamma(n)$ game.

Let, in $\Gamma(n)$ game for every $i \in N$ player's payoff function be m-dimensional vector-function $H_i: X \to R^m$ and comparison of their meanings with respect to the preference of the set of the situations takes place with lexicographic $\geq^L$ or $>^L$ relations. From two $a=(a_1, ..., a_m)$ and $b=(b_1, . . . , b_m)$ vectors $a >^L b$ if it fulfils one of the following conditions: 1. $a_1 > b_1$, 2. $a_1 = b_1$, $a_2 > b_2$, ..., m. $a_1 = b_1$, ..., $a_{m-1} = b_{m-1}$, $a_m > b_m$ and $a \geq^L b$, if $a >^L b$ or $a= b$. Such game is called lexicographic noncooperative game of n player and denote it $\Gamma^L(n)$. In such games Nashi's optimal principle – equilibrium situation may not exist neither in pure or mixed strategies [3-7].

In the paper we shall discuss the metaextensions – metagames of two players' lexicographic finite noncooperative games – lexicographic $\Gamma^L(2)$ bimatrix games. Let us define the main definitions connected with the informative and metastrategic extensions of two players of $\Gamma^L(2)$ game. Before it note that in the scalar noncooperative game the optimality of the situation in the sense of some principle becomes a version of the stability of the negotiation among the players [1]. In $\Gamma(n)$ game the purpose of the agreements: 1. If $i \in N$ player chooses $x_i \in X_i$ strategy, then $j \in N$ player

player will choose (or must choose) $x_j \in X_j$ strategy; 2. If i player in answer to the j player chooses $x_i \in X_i$ strategy, then k player must choose $x_k \in X_k$ strategy etc. Such conditional agreements are defined as metastrategies [1]. Thus metastrategies are defined as functions, they transform the set of one player's or coalition strategies into another player's set of strategies, or there are superpositions of such kinds of functions.

Let the two $N = \{1, 2\}$ players' finite lexicographic noncooperative game be

$$\Gamma^L = < \{1, 2\}, \{X, Y\}, \{H_1 = (H_1^1, ..., H_1^m), H_2 = (H_2^1, ..., H_2^m)\} >,$$

where X and Y are the sets of strategies of players 1 and 2, corresponding to it. Γ^L game is called lexicographic bimatrix game and it becomes by means of m scalar $\Gamma^1, \Gamma^2, \ldots, \Gamma^m$ bimatrix games $\Gamma^L = (\Gamma^1, \ldots, \Gamma^m)$ [6].

Let us discuss a simple example of the metastrategic extension for the bimatrix Γ^L game, where metastrategies are defined for the informative player.

Definition 1. In Γ^L game the player 2 is called informative, if its set Y of strategies has the following form $Y = Z^X$, or it consists of such functions which are defined on X and obtain their values from any Z set, and for any $x \in X, y \in Y$ strategies the players' payoff vector-functions have the following form $H_i(x,y) = (H_i^1, ..., H_i^m)(x, y(x))$, $i = 1, 2$.

Analogously, in Γ^L game player 1 is called informative, if $X = T^Y$ and for any $x \in X, y \in Y$ we have $H_i(x,y) = (H_i^1, ..., H_i^m)(x(y), y)$, $i = 1, 2$.

In Γ^L game player's information denotes its partner's preparation of choosing any kind of action. The players' payoffs do not depend on the plans of the informative players, but only on their choosing activities.

Definition 2. In conditions of Γ^L game the 1st player's metastrategy is called every other kind of function X^Y: $Y \to X$, and 2nd player's metastrategy is called $Y^X: X \to Y$ kind of function (thus X^Y is the 1st player's answer to the 2nd player's strategy and Y^X vice versa).

For the Γ^L game let us discuss the following two lexicographic bimatrix games for one and the same players'

$$\Gamma_1{}^L = < \{1, 2\}, \{X, Y^X\}, \{U_1 = (U_1^1, ..., U_1^m), U_2 = (U_2^1, ..., U_2^m)\} >,$$

$$\Gamma_2{}^L = < \{1, 2\}, \{X^Y, Y\}, \{V_1 = (V_1^1, ..., V_1^m), V_2 = (V_2^1, ..., V_2^m)\} >.$$

From these in $\Gamma_1{}^L$ game player 1st makes the first move and chooses $x \in X$ strategy, in $\Gamma_2{}^L$ game player 2 makes the first move and chooses $y \in Y$ strategy. In $\Gamma_1{}^L$ game for every $f : X \to Y$ function $U_i(x, f) = H_i(x, f(x))$, $i = 1, 2$; In $\Gamma_2{}^L$ game for $g : Y \to X$ function $V_i(g, y) = H_i(g(y), y)$, $i = 1, 2$.

Let us call such defined $\Gamma_1{}^L$ and $\Gamma_2{}^L$ games Γ^L game's first lexicographic metagames or Γ^L game's first lexicographic metastrategic extensions. The sets of strategies X and Y^X are the sets of metastrategies in the $\Gamma_1{}^L$ game and X^Y and Y are the sets of metastrategies in the $\Gamma_2{}^L$ game.

Theorem. *There always exists an equilibrium situation in the first lexicographic metastrategic extension of the lexicographic bimatrix* $\Gamma^L = (\Gamma^1, \ldots \Gamma^m)$ *game.*

Proof. Let us discuss the first lexicographic $\Gamma_1{}^L$ metagame of the bimatrix Γ^L game. In this game the set of metastrategies of the 1st player is the same as X set, and the set of metastrategies of the 2nd player consists of a finite number of functions $F = \{f\}$, of which each $f : X \to Y$. Let us choose any kind of strategy $x \in X$ of the 1st player, which has corresponding lines in the Γ^L and $\Gamma_1{}^L$ games, their elements are couples of R^m space vectors corresponding to $(H_1(x, y), H_2(x, y))$, $y \in Y$ and $(H_1(x, f(x)), H_2(x, f(x)))$, $f \in F$. Let us note that any two vectors of R^m are equal or between them one is more than the other lexicographically, i.e., by lexicographic binary relation. Therefore, we can always choose a lexicographically maximum element in the finite set of the vectors.

Let in the Γ^L game there correspond to each $x \in X$ strategy of the 1st player the 2nd player's such strategy $y(x)$ for which the lexicographic maximum $H_2(x, y(x)) = \overset{L}{\underset{y}{\max}} H_2(x, y)$ is achieved. It is obvious that the correspondence $x \to y(x)$ represents a function $f^* \in F$ and therefore from the latter equality it follows that

$$H_2(x, f^*(x)) \geq H_2(x, y), \ \forall x \in X, \ \forall y \in Y. \tag{1}$$

Now let us find such $x^* \in X$ that in the $(x^*, f^*(x^*)) = (x^*, f^*)$ situation the 1st player's payoff vectorial value in the $\Gamma_1{}^L$ game be lexicographically maximum

$$U_1(x^*,f^*) = H_1(x^*, f^*(x^*)) = \overset{L}{\max_x} H_1(x, f^*(x)). \qquad (2)$$

Let us show that the (x^*, f^*) situation is the equilibrium situation in the $\Gamma_1^{\,L}$ game. So, it is true that according to (2) for an $x \in X$ we have $U_1(x^*, f^*) \geq^L H_1(x, f^*(x)) = U_1(x, f^*)$.

From (1) it follows that for any $f \in F$

$$U_2(x^*, f) = H_2(x^*, f(x^*) \stackrel{L}{\leq} H_2(x^*, f^*(x^*)) = U_2(x^*, f^*).$$

Thus, $U_1(x^*, f^*) \geq^L U_1(x, f^*)$, $U_2(x^*, f^*) \geq^L U_2(x^*, f)$ for any $x \in X$, $f \in F$, it means that (x^*, f^*) is the equilibrium situation in the $\Gamma_1^{\,L}$ metagame.

It is proved analogously that in $\Gamma_2^{\,L}$ game there exists the equilibrium situation (g^*, y^*).

Note. If in the given bimatrix Γ^L game there exists an equilibrium situation (implying pure strategies), then in $\Gamma_1^{\,L}$ and $\Gamma_2^{\,L}$ metagames there may exist such equilibrium situations in which the players' payoffs are different from the corresponding payoffs in the equilibrium situation of the Γ^L game.

Let us discuss the possible variants with concrete examples.

Example 1. A lexicographic bimatrix game is given by matrix of payoff

	1	2
1	((1,0), (0,0))	((0,0), (1,1))
2	((0,0), (0,2))	((1,0), (0,1))

in which there is no equilibrium situation.

We consider that in bimatrix and matrix games the 1st player chooses the line of matrix of payoff, and the 2nd player chooses the column. In our example X=Y= {1, 2}.

Let us construct a $\Gamma_2^{\,L}$ metagame of the Γ^L game. Here the 2nd player will have the set of strategies Y= {1, 2} and the 1st player will have 4 metastrategies. Let us denote them in the form of a couple i_1/i_2 , where i_1 denotes the choice of own i_1 strategy in Γ^L game in the case of choosing the first strategy by the partner (2nd player), i_2 denotes the choice of i_2 strategy in the Γ^L game partner in the case of choosing the second strategy. Thus, the 1st player's metastrategies are 1) 1/1, 2) 1/2, 3) 2/1, 4) 2/2 in the Γ^L game. The values of the players' payoffs are given in the table:

	1	2
1/1	((1,0), (0,0))	((0,1), (1,1))
1/2	((1,0), (0,0))	((1,0), (0,1))*
2/1	((0,0), (0,2))	((0,1), (1,1))
2/2	((0,0), (0,2))	((1,0), (0,1))

In the given $\Gamma_2^{\,L}$ metagame the situation (1/2, 2) is an equilibrium situation and the players' payoffs in this situation are equal corresponding to (1,0) and (0,1):

$$(1,\ 0) \geq^L \begin{cases}(0,1), \\ (1,0),\end{cases} \qquad\qquad (0,\ 1) \geq^L (0,\ 0).$$

Example 2. **Lexicographic variant of the prisoners' dilemma**. Let us consider in every situation of the classical variant of the prisoners' dilemma the second more less important criterion as well as fine. Suppose, the lexicographic bimatrix $\Gamma^L = (\Gamma^1, \Gamma^2)$ game has the form:

	1	2
1	((−1,−2), (−1,−2))	((−10,−8), (0,0))
2	((0,0), (−10,−8))	((−8,−4), (−8,−4))*

Here the players' strategies sets X=Y={1, 2}= {1 –not to admit (cooperation), 2 – admit}. In Γ^L game as well as in the case of the scalar game, there exists only one equilibrium situation (2,2).

Let us construct a $\Gamma_1{}^L$ metagame of the Γ^L game. Here the 1st player has the set of strategies X={1, 2} and the 2nd player's set of metastrategies are {1/1, 1/2, 2/1, 2/2}. The values of players' payoffs meanings in the $\Gamma_1{}^L$ metagame have the following form:

	1/1	1/2	2/1	2/2
1	((−1,−2), (−1,−2))	((−1,−2), (−1,−2))	((−10,−8), (0,0))	((−10,−8), (0,0))
2	((0,0), (−10,−8))	((−8,−4), (−8,−4))	((0,0), (−10,−8))	((−8,−4), (−8,−4))*

(2, 2/2) is the equilibrium situation in the given $\Gamma_1{}^L$ game, in which the players' payoffs (-8,-4) and (-8,-4) respectively are equal to the players' payoffs in the equilibrium situation of the Γ^L game.

It should be noted that the classical variant of the "prisoners' dilemma" is checked in many social – psychological experiments, economic situations, in the politics of international relationship and in the theory of Law. The aim of the experiment was to create such conditions when it would be possible to reach the best result corresponding to the players' cooperation (not to admit the crime). To this end the prisoners' dilemma metagames in the case of scalar payoffs are discussed by S. Brams [8]. For this purpose let us discuss the lexicographic metagames of Γ^L variant of the prisoners' dilemma. As we see in $\Gamma_1{}^L$ metagame the best result ((-1,-2), (-1,-2)) hasn't been reached for both players.

Suppose that in the $\Gamma_1{}^L$ metagame the 1st player has the opportunity to ascertain what metastrategy the 2nd player will choose . Then the 1st player can determine his own metastrategies. In that way we get a $\Gamma_{12}{}^L$ metagame. As in $\Gamma_1{}^L$ game the 2nd player has four strategies and the 1st player has two strategies, therefore in $\Gamma_{12}{}^L$ game the 1st player will have 16 metastrategies, out of which each will be given in the following form *i/j/k/l* , where *i, j, k, l* = 1, 2. Here *i* denotes the choice of the 1st player by 2nd player in the process of choosing the 1st strategy, *j* – the choice of the 2nd player in the process of choosing the 2nd strategy etc. In this way from $\Gamma_1{}^L$ metagame we get a second metagame $\Gamma_{12}{}^L$ in which the players' payoffs have the form:

	1/1	1/2	2/1	2/2
1.1/1/1/1	((−1,−2), (−1,−2))	((−1,−2), (−1,−2))	((−10,−8), (0,0)	((−10,−8), (0,0)
2.1/1/1/2	((−1,−2), (−1,−2))	((−1,−2), (−1,−2))	((−10,−8), (0,0))	((−8,−4), (−8,−4))
3.1/1/2/1	((−1,−2), (−1,−2))	((−1,−2), (−1,−2))	((0,0), (−10,−8))	((−10,−8), (0,0))
4.1/2/1/1	((−1,−2), (−1,−2))	((−8,−4), (−8,−4))	((−10,−8), (0,0))	((−10,−8), (0,0))
5.2/1/1/1	((0,0), (−10,−8))	((−1,−2), (−1,−2))	((−10,−8), (0,0))	((−10,−8), (0,0))
6.1/1/2/2	((−1,−2), (−1,−2))	((−1,−2), (−1,−2))*	((0,0), (−10,−8))	((−8,−4), (−8,−4))
7.1/2/1/2	((−1,−2), (−1,−2))	((−8,−4), (−8,−4))	((−10,−8), (0,0))	((−8,−4), (−8,−4))
8.2/1/1/2	((0,0), (−10,−8))	((−1,−2), (−1,−2))	((−10,−8), (0,0))	((−8,−4), (−8,−4))
9.1/2/2/1	((−1,−2), (−1,−2))	((−8,−4), (−8,−4))	((0,0), (−10,−8))	((−10,−8), (0,0))
10.2/1/2/1	((0,0), (−10,−8))	((−1,−2), (−1,−2))	((0,0), (−10,−8))	((−10,−8), (0,0))
11.2/2/1/1	((0,0), (−10,−8))	((−8,−4), (−8,−4))	((−10,−8), (0,0))	((−10,−8), (0,0))
12.1/2/2/2	((−1,−2), (−1,−2))	((−8,−4), (−8,−4))	((0,0), (−10,−8))	((−8,−4), (−8,−4))
13.2/1/2/2	((0,0), (−10,−8))	((−1,−2), (−1,−2))*	((0,0), (−10,−8))	((−8,−4), (−8,−4))
14.2/2/1/2	((0,0), (−10,−8))	((−8,−4)(−8,−4))	((−10,−8), (0,0))	((−8,−4), (−8,−4))
15.2/2/2/1	((0,0), (−10,−8))	((−8,−4), (−8,−4))	((0,0), (−10,−8))	((−10,−8), (0,0))
16.2/2/2/2	((0,0), (−10,−8))	((−8,−4), (−8,−4))	((0,0), (−10,−8))	((−8,−4), (−8,−4))*

There are three equilibrium situations in the $\Gamma_{12}{}^L$ game: (1/1/2/2, 1/2), (2/1/2/2, 1/2), (2/2/2/2, 2/2), two of them have one and the same best result ((-1,-2), (-1,-2)), which cannot be achieved by the equilibrium situation in the Γ^L and $\Gamma_1{}^L$ games. These payoffs also correspond to the optimal situation by Pareto's sense .

This example shows that the result of N. Howard's [2] construction, respectively to the above, there is a situation which will be one and the same time equilibrium and optimal according to Pareto in the metaextensions of two players' finite noncooperative game, it must be fair for the lexicographic games. We consider that, using the same construction, it would be possible to prove this and other famous classical results for *n* player's lexicographic noncooperative games. However, feature characteristic of lexicograply may take place in this kind of case.

Example 3. Let us discuss a lexicographic matrix $\Gamma^L = (\Gamma^1, \Gamma^2)$ game by the matrix of payoff

	1	2
1	(1,0)	(0,0)
2	(0,0)	(0,1)

.

In this game as already shown by P. Fischburn [6], there is no saddle point in the pure and mixed strategies. Let us discuss a $\Gamma_1{}^L$ metagame of the Γ^L game:

	1/1	1/2	2/1	2/2
1	(1,0)	(1,0)	$(0,0)^*$	(0,0)
2	(0,0)	(0,1)	$(0,0)^*$	(0,1)

.

Here we have 2 saddle points (1, 2/1) and (2, 2/1), where the 1st player's payoffs are the same and equal (0,0).

მათემატიკა

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ცნობილია, რომ სასრულ ლექსიკოგრაფიულ არაკოოპერატიულ თამაშში წონასწორობის სიტუაცია შეიძლება არ არსებობდეს არც წმინდა და არც შერეულ სტრატეგიებში. ნაშრომში განხილულია ორი მოთამაშის სასრული ლექსიკოგრაფიული არაკოოპერატიული თამაშის – ლექსიკოგრაფიული ბიმატრიცული თამაშის ინფორმაციული და მეტასტრატეგიული გაფართოებები – ლექსიკოგრაფიული ბიმატრიცული მეტათამაშები. დამტკიცებულია, რომ პირველ ბიმატრიცულ მეტათამაშებში ყოველთვის არსებობს წონასწორობის სიტუაცია. მაგალითზე ნაჩვენებია, რომ მეორე ბიმატრიცულ მეტაგაფართოებაში არსებობს ისეთი წონასწორობის სიტუაციაც, რომელიც ამავე დროს არის ოპტიმალური პარეტოს აზრით.

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