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Mathematical Physics

Construction and Investigation of Hierarchical Models for Thermoelastic Prismatic Shells

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ABSTRACT. In the present paper an initial boundary value problem for thermoelastic prismatic shells is considered. Three-dimensional dynamical problem for prismatic shell with surface forces given along the upper and the lower faces of the shell is reduced to hierarchy of two-dimensional problems. The obtained problems are investigated in suitable function spaces, the convergence of the sequence of vector-functions of three space variables, restored from the solutions of two-dimensional problems to the solution of the original three-dimensional problem, is proved and the rate of approximation is estimated. *© 2008 Bull. Georg. Natl. Acad. Sci.*

Key words: *dynamical problems for thermoelastic prismatic shells, Fourier-Legendre series.*

Two-dimensional and one-dimensional approximations of three-dimensional boundary and initial-boundary value problems are often used when constructing mathematical models of complicated building and various engineering structures. One of the methods of constructing two-dimensional models for linearly elastic prismatic shells was suggested by I. Vekua in the paper [1]. In this paper Vekua considered a three-dimensional linear model of an elastic prismatic shell and expanding components of the displacement vector-function into orthogonal Fourier-Legendre series with respect to the variable of shell thickness the hierarchy of differential two-dimensional models was obtained. Various two-dimensional models constructed by Vekua were collected in his monograph [2]. The estimates of accuracy for the two-dimensional hierarchical models constructed in [1] first were obtained in the spaces of classical regular functions in the paper [3], and the reduced two-dimensional models for thin shallow shells constructed in [2] were investigated in Sobolev spaces in [4]. Later on, Vekua's reduction method considered in [1, 2] and its generalizations were studied in the papers [5-12].

The present paper is devoted to the construction and investigation of two-dimensional hierarchical models of prismatic shell, taking into account thermal properties of elastic solid, which were mainly neglected when constructing hierarchical models of elastic bodies. We consider the variational formulation of three-dimensional initial-boundary value problem for linearly thermoelastic prismatic shell and construct its two-dimensional hierarchical model in Sobolev spaces, when the temperature equals zero along the boundary of the body and surface forces are given along the upper and the lower faces of the shell. We investigate the existence and uniqueness of solutions of the reduced two-dimensional problems in suitable weighted Sobolev spaces. Moreover, we prove the convergence of the sequence of vector-functions of three space variables to the solution of the original three-dimensional problem and if it possesses additional regularity, we estimate the rate of convergence.

Let $\Omega \subset \mathbf{R}^p, p \geq 1$ be a bounded domain with Lipschitz boundary. $L^2(\Omega)$ denotes the space of square-integrable functions in Ω in the Lebesgue sense. $W^{k,2}(\Omega) = H^k(\Omega)$, $k \geq 1$ is the Sobolev space of order k based on $L^2(\Omega)$,

$\mathbf{H}^k(\Omega)=\left[H^k(\Omega)\right]^3$, $\mathbf{L}^2(\Omega)=\left[L^2(\Omega)\right]^3$. We denote by $H_0^k(\Omega)$ the closure of the set $D(\Omega)$ of infinitely differentiable functions with compact support in Ω in the space $H^k(\Omega)$, $k\geq 1$. For any Banach space X, $C^0([0,T];X)$ denotes the space of continuous vector-functions on $[0,T]$ with values in X, $L^2(0,T;X)$ is the space of such vector-functions $g:(0,T)\to X$ that $\|g(t)\|_X\in L^2(0,T)$. We denote by $g'=dg/dt$ the generalized derivative of $g\in L^2(0,T;X)$.

Let us consider heat conduction process in prismatic shell with thickness vanishing on a part of its boundary, i.e. when the prismatic shell is a Lipschitz domain Ω which has the following form

$$\Omega=\left\{(x_1,x_2,x_3)\in\mathbf{R}^3;\, h^-(x_1,x_2)<x_3<h^+(x_1,x_2),\quad (x_1,x_2)\in\omega\right\},$$

where $\omega\subset\mathbf{R}^2$ is a two-dimensional bounded Lipschitz domain with boundary $\partial\omega$, $h^\pm\in C^0(\overline{\omega})\cap Lip(\omega\cup\widetilde{\gamma})$ are Lipschitz continuous in ω and on $\widetilde{\gamma}\subset\partial\omega$, $h^+(x_1,x_2)>h^-(x_1,x_2)$, for $(x_1,x_2)\in\omega\cup\widetilde{\gamma}$, $\widetilde{\gamma}\subset\partial\omega$ is a Lipschitz curve, $h^+(x_1,x_2)=h^-(x_1,x_2)$, for $(x_1,x_2)\in\partial\omega\setminus\widetilde{\gamma}$. The upper and the lower faces of Ω, defined by the equations $x_3=h^+(x_1,x_2)$ and $x_3=h^-(x_1,x_2)$, $(x_1,x_2)\in\omega$, we denote by Γ^+ and Γ^-, respectively, and the lateral face, where the thickness of Ω is positive, we denote by $\widetilde{\Gamma}=\partial\Omega\setminus(\overline{\Gamma^+\cup\Gamma^-})=\{(x_1,x_2,x_3)\in\mathbf{R}^3;h^-(x_1,x_2)<x_3<h^+(x_1,x_2),$ $(x_1,x_2)\in\widetilde{\gamma}\}$.

We assume that the prismatic shell consists of homogeneous, isotropic thermoelastic material with Lamé constants λ, μ, density ρ and thermoelastic constants γ, η, χ, which define the mechanical properties of the material. The applied body force density we denote by $\mathbf{f}=(f_i):\Omega\times(0,T)\to\mathbf{R}^3$ and the density of heat sources we denote by $f^\theta:\Omega\times(0,T)\to\mathbf{R}$. The temperature θ vanishes along the boundary $\Gamma=\partial\Omega$ of the domain Ω. The prismatic shell is clamped along a part $\widetilde{\Gamma}_0=\{(x_1,x_2,x_3)\in\mathbf{R}^3;h^-(x_1,x_2)<x_3<h^+(x_1,x_2),\ (x_1,x_2)\in\widetilde{\gamma}_0\}$, $\widetilde{\gamma}_0\subset\widetilde{\gamma}$, of the lateral face $\widetilde{\Gamma}$ and on the remaining part $\Gamma_1=\Gamma\setminus\overline{\widetilde{\Gamma}_0}$ of the boundary surface force with density $\mathbf{g}=(g_i):\Gamma_1\times(0,T)\to\mathbf{R}^3$ is given. The dynamical linear three-dimensional model of stress-strain state of thermoelastic body in differential form is given by

$$\frac{\partial^2u_i}{\partial t^2}-\frac{1}{\rho}\sum_{j=1}^3\frac{\partial}{\partial x_j}\left(\lambda\sum_{p=1}^3 e_{pp}(\mathbf{u})\delta_{ij}+2\mu e_{ij}(\mathbf{u})-\gamma\theta\delta_{ij}\right)=f_i\ \text{ in }\Omega,\ i=1,2,3, \tag{1}$$

$$\frac{\partial\theta}{\partial t}-\chi\sum_{j=1}^3\frac{\partial^2\theta}{\partial x_j^2}+\chi\eta\frac{\partial}{\partial t}\sum_{p=1}^3 e_{pp}(\mathbf{u})=f^\theta\ \text{ in }\Omega, \tag{2}$$

$$\sum_{j=1}^3\left(\lambda\sum_{p=1}^3 e_{pp}(\mathbf{u})\delta_{ij}+2\mu e_{ij}(\mathbf{u})-\gamma\theta\delta_{ij}\right)\nu_j=g_i\ \text{ on }\Gamma_1,\ \ \mathbf{u}(0)=\mathbf{0}\ \text{ on }\widetilde{\Gamma}_0,\ \theta=0\ \text{ on }\Gamma, \tag{3}$$

$$u_i(x,0)=u_{0i}(x),\ \frac{\partial u_i}{\partial t}(x,0)=u_{1i}(x),\ \theta(x,0)=\theta_0(x)\ \text{ in }\Omega,\ i=1,2,3, \tag{4}$$

where $\mathbf{u}=(u_i):\Omega\times(0,T)\to\mathbf{R}^3$ is the displacement vector-function of thermoelastic body, $\theta:\Omega\times(0,T)\to\mathbf{R}$ is the temperature distribution. If we multiply equations (1) by smooth enough functions v_i, which vanish on $\widetilde{\Gamma}_0$, multiply equation (2) by smooth enough function φ vanishing on Γ, integrate the obtained equations over the domain Ω and apply integration by parts, taking into account boundary conditions (3), we obtain

$$\sum_{i=1}^{3}\int_{\Omega}\frac{\partial^2 u_i}{\partial t^2}v_i dx+\frac{1}{\rho}\int_{\Omega}\left(\lambda\sum_{p=1}^{3}e_{pp}(\mathbf{u})\sum_{q=1}^{3}e_{qq}(\mathbf{v})+2\mu\sum_{i,j=1}^{3}e_{ij}(\mathbf{u})e_{ij}(\mathbf{v})\right)dx+$$

$$+\frac{\gamma}{\rho}\sum_{i=1}^{3}\int_{\Omega}\frac{\partial\theta}{\partial x_i}v_i dx=\sum_{i=1}^{3}\int_{\Omega}f_i(x)v_i(x)dx+\frac{1}{\rho}\sum_{i=1}^{3}\int_{\Gamma_1}g_i v_i d\Gamma, \tag{5}$$

$$\int_{\Omega}\frac{\partial\theta}{\partial t}\varphi dx+\chi\sum_{j=1}^{3}\int_{\Omega}\frac{\partial\theta}{\partial x_j}\frac{\partial\varphi}{\partial x_j}dx-\chi\eta\sum_{p=1}^{3}\int_{\Omega}\frac{\partial u_p}{\partial t}\frac{\partial\varphi}{\partial x_p}dx=\int_{\Omega}f^{\theta}(x)\varphi(x)dx, \tag{6}$$

for all $\mathbf{v}=(v_i)$, which are smooth enough and equal to zero on $\tilde{\Gamma}_0$, and for all sufficiently regular φ vanishing on the boundary Γ. Note that if $\mathbf{u}=(u_i)$ and θ are solutions of the equations (5) and (6) and are smooth enough, then they also satisfy differential equations (1), (2) and boundary conditions (3). So, the problem (1)-(4) is equivalent to the problem (5), (6), (4), which can be used to define the weak solution of the initial boundary value problem for thermoelastic prismatic shell.

Hereafter we consider the following variational formulation of the three-dimensional initial boundary value problem (1)-(4): find $\mathbf{u}\in C^0([0,T];\mathbf{V}(\Omega))$, $\mathbf{u}'\in C^0([0,T];\mathbf{L}^2(\Omega))$, $\theta\in L^2(0,T;H_0^1(\Omega))$, which satisfies the following equations in the sense of distributions in $(0,T)$

$$\frac{d}{dt}(\mathbf{u}'(.),\mathbf{v})_{\mathbf{L}^2(\Omega)}+a(\mathbf{u}(.),\mathbf{v})+\frac{\gamma}{\rho}\sum_{i=1}^{3}(\frac{\partial\theta}{\partial x_i},v_i)_{L^2(\Omega)}=(\mathbf{f},\mathbf{v})_{\mathbf{L}^2(\Omega)}+\frac{1}{\rho}(\mathbf{g},\mathbf{v})_{\mathbf{L}^2(\Gamma_1)},\ \ \forall\mathbf{v}\in\mathbf{V}(\Omega), \tag{7}$$

$$\frac{d}{dt}(\theta(.),\varphi)_{L^2(\Omega)}+a^{\theta}(\theta(.),\varphi)-\chi\eta\sum_{p=1}^{3}(u'_p,\frac{\partial\varphi}{\partial x_p})_{L^2(\Omega)}=(f^{\theta},\varphi)_{L^2(\Omega)},\ \ \forall\varphi\in H_0^1(\Omega), \tag{8}$$

together with the initial conditions

$$\mathbf{u}(0)=\mathbf{u}_0,\ \ \mathbf{u}'(0)=\mathbf{u}_1,\ \ \theta(0)=\theta_0, \tag{9}$$

where $\mathbf{u}_0$, $\mathbf{u}_1$ are the initial displacement and velocity vector-functions, θ_0 is the initial distribution of temperature, $\mathbf{u}_0\in\mathbf{V}(\Omega)=\{\mathbf{v}\in\mathbf{H}^1(\Omega);\mathbf{tr}(\mathbf{v})=\mathbf{0}\ \text{ on } \tilde{\Gamma}_0\}$, $\mathbf{tr}$ is the trace operator from $\mathbf{H}^1(\Omega)$ to $\mathbf{H}^{1/2}(\Gamma)$,

$$a(\tilde{\mathbf{v}},\mathbf{v})=\frac{1}{\rho}\int_{\Omega}\left(\lambda\sum_{p=1}^{3}e_{pp}(\tilde{\mathbf{v}})\sum_{q=1}^{3}e_{qq}(\mathbf{v})+2\mu\sum_{i,j=1}^{3}e_{ij}(\tilde{\mathbf{v}})e_{ij}(\mathbf{v})\right)dx,\qquad\forall\mathbf{v},\tilde{\mathbf{v}}\in\mathbf{V}(\Omega),$$

$$a^{\theta}(\tilde{\varphi},\varphi)=\chi\int_{\Omega}\sum_{i=1}^{3}\frac{\partial\tilde{\varphi}}{\partial x_i}\frac{\partial\varphi}{\partial x_i}dx,\qquad\forall\varphi,\tilde{\varphi}\in H_0^1(\Omega),\ \ e_{ij}(\mathbf{v})=1/2(\partial_i v_j+\partial_j v_i),\ i,j=\overline{1,3},$$

$(.,.)_{\mathbf{L}^2(\Omega)}$, $(.,.)_{L^2(\Omega)}$ and $(.,.)_{\mathbf{L}^2(\Gamma_1)}$ are scalar products in the spaces $\mathbf{L}^2(\Omega)$, $L^2(\Omega)$ and $\mathbf{L}^2(\Gamma_1)$, respectively.

In order to construct the hierarchy of two-dimensional models let us consider the subspaces $\mathbf{V_N}(\Omega)$ and $\mathbf{H_N}(\Omega)$ of $\mathbf{V}(\Omega)$ and $\mathbf{L}^2(\Omega)$, respectively, $\mathbf{N}=(N_1,N_2,N_3)$, consisting of vector-functions whose components are polynomials with respect to the variable x_3,

$$\mathbf{v_N}=(v_{\mathbf{N}i}),\ \ v_{\mathbf{N}i}=\sum_{r_i=0}^{N_i}\frac{1}{h}(r_i+\frac{1}{2})\overset{r_i}{v}_{\mathbf{N}i}P_{r_i}(z),\quad \overset{r_i}{v}_{\mathbf{N}i}\in L^2(\omega),\quad 0\le r_i\le N_i,\quad i=1,2,3,$$

where $z=\dfrac{x_3-\bar{h}}{h}$, $h=\dfrac{h^+-h^-}{2}$, $\bar{h}=\dfrac{h^++h^-}{2}$. We also consider the subspaces $V^{\theta}_{N_\theta}(\Omega)$ and $H^{\theta}_{N_\theta}(\Omega)$ of $H^1_0(\Omega)$ and $L^2(\Omega)$, respectively, which consist of the following functions

$$\varphi_{N_\theta}=\sum_{r=0}^{N_\theta}\frac{1}{h}(r+\frac{1}{2})\overset{r}{\varphi}_{N_\theta}P_r(z)-\frac{1}{2}\sum_{r=0}^{N_\theta}\frac{1}{h}(1-(-1)^{r+N_\theta})(r+\frac{1}{2})\overset{r}{\varphi}_{N_\theta}P_{N_\theta+1}(z)-$$

$$-\frac{1}{2}\sum_{r=0}^{N_\theta}\frac{1}{h}(1+(-1)^{r+N_\theta})(r+\frac{1}{2})\overset{r}{\varphi}_{N_\theta}P_{N_\theta+2}(z),\quad \overset{r}{\varphi}_{N_\theta}\in L^2(\omega),\ r=\overline{0,N_\theta}.$$

Note that the functions h^+ and h^- are Lipschitz continuous in ω and hence, due to Rademacher's theorem [13], h^+ and h^- are differentiable almost everywhere in ω and $\partial_\alpha h^\pm\in L^\infty(\omega^*)$ for all subdomains ω^*, $\overline{\omega^*}\subset\omega$, $\alpha=1,2$. Therefore, the positiveness of h in ω implies that for any vector-function $\mathbf{v_N}=(v_{\mathbf{N}i})_{i=1}^3\in\mathbf{V_N}(\Omega)$ the corresponding functions $\overset{r_i}{v}_{\mathbf{N}i}\in H^1(\omega^*)$ for all ω^*, $\overline{\omega^*}\subset\omega$, i.e. $\overset{r_i}{v}_{\mathbf{N}i}\in H^1_{loc}(\omega)$, $0\le r_i\le N_i$, $i=1,2,3$. Similarly, for all functions $\varphi_{N_\theta}\in V^{\theta}_{N_\theta}(\Omega)$, the functions $\overset{r}{\varphi}_{N_\theta}$ of two space variables in the expressions of φ_{N_θ} belong to $H^1(\omega^*)$, $\overline{\omega^*}\subset\omega$, i.e. $\overset{r}{\varphi}_{N_\theta}\in H^1_{loc}(\omega)$, $r=\overline{0,N_\theta}$. Moreover, the norms $\|\cdot\|_{\mathbf{H}^1(\Omega)}$ and $\|\cdot\|_{H^1(\Omega)}$ in the spaces $\mathbf{H}^1(\Omega)$ and $H^1(\Omega)$ define weighted norms $\|\cdot\|_*$ and $\|\cdot\|_{\theta*}$ of vector-functions $v_{\mathbf{N}}\in[H^1_{loc}(\omega)]^{N_{1,2,3}}$, $N_{1,2,3}=N_1+N_2+N_3+3$, with components $\overset{r_i}{v}_{\mathbf{N}i}$, $\vec{v}_{\mathbf{N}}=(\overset{r_i}{v}_{\mathbf{N}i})$, and $\vec{\varphi}_{N_\theta}\in[H^1_{loc}(\omega)]^{N_\theta+1}$, with components $\overset{r}{\varphi}_{N_\theta}$, $\vec{\varphi}_{N_\theta}=(\overset{r}{\varphi}_{N_\theta})$, such that $\|\vec{v}_{\mathbf{N}}\|_*=\|\mathbf{v_N}\|_{\mathbf{H}^1(\Omega)}$ and $\|\vec{\varphi}_{N_\theta}\|_{\theta*}=\|\varphi_{N_\theta}\|_{H^1(\Omega)}$. Using the properties of the Legendre polynomials [14], we obtain explicit expressions of the norms $\|\cdot\|_*$ and $\|\cdot\|_{\theta*}$,

$$\|\vec{v}_{\mathbf{N}}\|_*^2=\sum_{i=1}^{3}\sum_{r_i=0}^{N_i}(r_i+\frac{1}{2})\left[\left\|\sum_{s_i=r_i}^{N_i}(s_i+\frac{1}{2})(1-(-1)^{r_i+s_i})h^{-3/2}\overset{s_i}{v}_{\mathbf{N}i}\right\|^2_{L^2(\omega)}+\left\|h^{-1/2}\overset{r_i}{v}_{\mathbf{N}i}\right\|^2_{L^2(\omega)}+\right.$$

$$\left.+\sum_{\alpha=1}^{2}\left\|\sum_{s_i=r_i+1}^{N_i}(s_i+\frac{1}{2})(\partial_\alpha h^+-(-1)^{r_i+s_i}\partial_\alpha h^-)h^{-3/2}\overset{s_i}{v}_{\mathbf{N}i}-h^{-1/2}\partial_\alpha\overset{r_i}{v}_{\mathbf{N}i}+(r_i+1)h^{-3/2}\partial_\alpha h\overset{r_i}{v}_{\mathbf{N}i}\right\|^2_{L^2(\omega)}\right],$$

$$\|\vec{\varphi}_{N_\theta}\|_{\theta*}^2=\sum_{r=0}^{N_\theta+2}(r+\frac{1}{2})\left[\left\|\sum_{s=r}^{N_\theta+2}(s+\frac{1}{2})(1-(-1)^{r+s})h^{-3/2}\overset{s}{\varphi}_{N_\theta}\right\|^2_{L^2(\omega)}+\left\|h^{-1/2}\overset{r}{\varphi}_{N_\theta}\right\|^2_{L^2(\omega)}+\right.$$

$$\left.+\sum_{\alpha=1}^{2}\left\|\sum_{s=r+1}^{N_\theta+2}(s+\frac{1}{2})(\partial_\alpha h^+-(-1)^{r+s}\partial_\alpha h^-)h^{-3/2}\overset{s}{\varphi}_{N_\theta}-h^{-1/2}\partial_\alpha\overset{r}{\varphi}_{N_\theta}+(r+1)h^{-3/2}\partial_\alpha h\overset{r}{\varphi}_{N_\theta}\right\|^2_{L^2(\omega)}\right],$$

where we assume that the sum with the lower limit greater than the upper one equals zero,

$$\overset{N_\theta+1}{\varphi}_{N_\theta}=-\frac{1}{2N_\theta+3}\sum_{r=0}^{N_\theta}(1-(-1)^{r+N_\theta})\left(r+\frac{1}{2}\right)\overset{r}{\varphi}_{N_\theta},\quad \overset{N_\theta+2}{\varphi}_{N_\theta}=-\frac{1}{2N_\theta+5}\sum_{r=0}^{N_\theta}(1+(-1)^{r+N_\theta})\left(r+\frac{1}{2}\right)\overset{r}{\varphi}_{N_\theta}.$$

For components $\overset{r_i}{v}_{\mathbf{N}i}$ and $\overset{r}{\varphi}_{N_\theta}$ of $v_{\mathbf{N}}\in[H^1_{loc}(\omega)]^{N_{1,2,3}}$ and $\bar{\varphi}_{N_\theta}\in[H^1_{loc}(\omega)]^{N_\theta+1}$, which possess the properties

$\left\|\vec{v}_{\mathbf{N}}\right\|_{*}<\infty$ and $\left\|\vec{\varphi}_{N_\theta}\right\|_{\theta *}<\infty$ we can define the trace on $\tilde{\gamma}$. Indeed, the corresponding vector-function of three space variables $\mathbf{v_N}=(v_{\mathbf{N}i})_{i=1}^{3}$ and function φ_{N_θ} belong to the space $\mathbf{V_N}(\Omega)\subset\mathbf{H}^1(\Omega)$ and $V^{\theta}_{N_\theta}(\Omega)\subset H^1(\Omega)$, respectively. Consequently, applying the trace operator $tr: H^1(\Omega)\to H^{1/2}(\Gamma)$ on the space $H^1(\Omega)$, we define the traces on $\tilde{\gamma}$ for $\overset{r_i}{v}_{\mathbf{N}i}$ and $\overset{r}{\varphi}_{N_\theta}$,

$$tr_{\tilde{\gamma}}(\overset{r_i}{v}_{\mathbf{N}i})=\int_{h^-}^{h^+} tr(v_{\mathbf{N}i})|_{\tilde{\Gamma}}\, P_{r_i}(z)dx_3,\quad tr_{\tilde{\gamma}}(\overset{r}{\varphi}_{N_\theta})=\int_{h^-}^{h^+} tr(\varphi_{N_\theta})|_{\tilde{\Gamma}}\, P_r(z)dx_3,\quad r_i=\overline{0,N_i},\ i=\overline{1,3}\ ,\ r=\overline{0,N_\theta}.$$

Since the vector-functions $\mathbf{v_N}=(v_{\mathbf{N}i})$ from the subspaces $\mathbf{V_N}(\Omega)$ and $\mathbf{H_N}(\Omega)$, and the functions φ_{N_θ} from $V^{\theta}_{N_\theta}(\Omega)$ and $H^{\theta}_{N_\theta}(\Omega)$ are defined by functions $\overset{r_i}{v}_{\mathbf{N}i}$ and $\overset{r}{\varphi}_{N_\theta}$ of two space variables, therefore considering the original three-dimensional problem (7)-(9) on these subspaces, we obtain the following hierarchy of two-dimensional problems: find $\vec{w}_{\mathbf{N}}\in C^0([0,T];\vec{V}_{\mathbf{N}}(\omega))$, $\vec{w}'_{\mathbf{N}}\in C^0([0,T];\vec{H}_{\mathbf{N}}(\omega))$, $\vec{\zeta}_{N_\theta}\in L^2(0,T;\vec{V}^{\theta}_{N_\theta}(\omega))$, which satisfy the following equations in the sense of distributions in $(0,T)$

$$\frac{d}{dt}R_{\mathbf{N}}(\vec{w}'_{\mathbf{N}}(.),\vec{v}_{\mathbf{N}})+a_{\mathbf{N}}(\vec{w}_{\mathbf{N}}(.),\vec{v}_{\mathbf{N}})+\frac{\gamma}{\rho}b_{\mathbf{N}N_\theta}(\vec{\zeta}_{N_\theta}(.),\vec{v}_{\mathbf{N}})=L_{\mathbf{N}}(\vec{v}_{\mathbf{N}}),\qquad \forall\vec{v}_{\mathbf{N}}\in\vec{V}_{\mathbf{N}}(\omega), \tag{10}$$

$$\frac{d}{dt}R^{\theta}_{N_\theta}(\vec{\zeta}_{N_\theta}(.),\vec{\varphi}_{N_\theta})+a^{\theta}_{N_\theta}(\vec{\zeta}_{N_\theta}(.),\vec{\varphi}_{N_\theta})-\chi\eta b^{\theta}_{\mathbf{N}N_\theta}(\vec{w}'_{\mathbf{N}}(.),\vec{\varphi}_{N_\theta})=L^{\theta}_{N_\theta}(\vec{\varphi}_{N_\theta}),\quad \forall\vec{\varphi}_{N_\theta}\in\vec{V}^{\theta}_{N_\theta}(\omega), \tag{11}$$

together with the initial conditions

$$\vec{w}_{\mathbf{N}}(0)=\vec{w}_{\mathbf{N}0},\qquad \vec{w}'_{\mathbf{N}}(0)=\vec{w}_{\mathbf{N}1},\qquad \vec{\zeta}_{N_\theta}(0)=\vec{\zeta}_{N_\theta 0}, \tag{12}$$

where $\vec{w}_{\mathbf{N}0}\in\vec{V}_{\mathbf{N}}(\omega)=\left\{\vec{v}_{\mathbf{N}}=(\overset{r_i}{v}_{\mathbf{N}i})\in[H^1_{loc}(\omega)]^{N_{1,2,3}};\left\|\vec{v}_{\mathbf{N}}\right\|_{*}<\infty, tr_{\tilde{\gamma}}(\overset{r_i}{v}_{\mathbf{N}i})=0 \text{ on } \tilde{\gamma}_0, r_i=\overline{0,N_i}, i=\overline{1,3}\right\}$,

$$\vec{w}_{\mathbf{N}1}\in\vec{H}_{\mathbf{N}}(\omega)=\left\{\vec{v}_{\mathbf{N}}=(\overset{r_i}{v}_{\mathbf{N}i})\in[L^2(\omega)]^{N_{1,2,3}};\left\|\vec{v}_{\mathbf{N}}\right\|^2_{\vec{H}_{\mathbf{N}}(\omega)}=\sum_{i=1}^{3}\sum_{r_i=0}^{N_i}\left\|h^{-1/2}\overset{r_i}{v}_{\mathbf{N}i}\right\|^2_{L^2(\omega)}<\infty\right\},$$

$$\vec{V}^{\theta}_{N_\theta}(\omega)=\left\{\vec{\varphi}_{N_\theta}=(\overset{r}{\varphi}_{N_\theta})\in[H^1_{loc}(\omega)]^{N_\theta+1};\left\|\vec{\varphi}_{N_\theta}\right\|_{\theta *}<\infty, tr_{\tilde{\gamma}}(\overset{r}{\varphi}_{N_\theta})\equiv 0, r=\overline{0,N}\right\}\ ,$$

$\vec{\zeta}_{N_\theta 0}\in\vec{H}^{\theta}_{N_\theta}(\omega)=\left\{\vec{\varphi}_{N_\theta}=(\overset{r}{\varphi}_{N_\theta})\in[L^2(\omega)]^{N_\theta+1};\left\|\vec{\varphi}_{N_\theta}\right\|_{\vec{H}^{\theta}_{N_\theta}(\omega)}=\sum_{r=0}^{N_\theta}\left\|h^{-1/2}\overset{r}{\varphi}_{N_\theta}\right\|^2_{L^2(\omega)}<\infty\right\}$, the bili-near forms $R_{\mathbf{N}}$, $R^{\theta}_{N_\theta}$, $a_{\mathbf{N}}$, $a^{\theta}_{N_\theta}$, $b_{\mathbf{N}N_\theta}$, $b^{\theta}_{\mathbf{N}N_\theta}$ are defined by the corresponding forms in the left-hand sides of the equations (7), (8) and taking into account the properties of the Legendre polynomials, we have

$$R_{\mathbf{N}}(\vec{y}_{\mathbf{N}},\vec{v}_{\mathbf{N}})=\sum_{i=1}^{3}\sum_{r_i=0}^{N_i}\left(r_i+\frac{1}{2}\right)\int_{\omega}\frac{1}{h}\overset{r_i}{y}_{\mathbf{N}i}\overset{r_i}{v}_{\mathbf{N}i}\,d\omega,\quad R^{\theta}_{N_\theta}(\vec{\psi}_{N_\theta},\vec{\varphi}_{N_\theta})=\sum_{r=0}^{N_\theta+2}\left(r+\frac{1}{2}\right)\int_{\omega}\frac{1}{h}\overset{r}{\psi}_{N_\theta}\overset{r}{\varphi}_{N_\theta}\,d\omega,$$

$$a_{\mathbf{N}}(\vec{y}_{\mathbf{N}},\vec{v}_{\mathbf{N}})=\frac{1}{\rho}\sum_{r=0}^{N_{\max}}\left(r+\frac{1}{2}\right)\int_{\omega}\frac{1}{h}\left(\lambda\sum_{p=1}^{3}\overset{r}{e}_{pp}(\vec{y}_{\mathbf{N}})\sum_{q=1}^{3}\overset{r}{e}_{qq}(\vec{v}_{\mathbf{N}})+2\mu\sum_{i,j=1}^{3}\overset{r}{e}_{ij}(\vec{y}_{\mathbf{N}})\overset{r}{e}_{ij}(\vec{v}_{\mathbf{N}})\right)d\omega,$$

$$a^{\theta}_{N_\theta}(\vec{\psi}_{N_\theta},\vec{\varphi}_{N_\theta}) = \chi \sum_{r=0}^{N_\theta+2}\left(r+\frac{1}{2}\right)\int_{\omega}\left[\frac{1}{h^3}\left(\sum_{s=r}^{N_\theta+2}(s+\frac{1}{2})\overset{s}{\psi}_{N_\theta}(1-(-1)^{r+s})\right)\left(\sum_{\hat{s}=r}^{N_\theta+2}(\hat{s}+\frac{1}{2})\overset{\hat{s}}{\varphi}_{N_\theta}(1-(-1)^{r+\hat{s}})\right)+\right.$$

$$+\sum_{\alpha=1}^{2}\frac{1}{h}\left(\partial_\alpha \overset{r}{\psi}_{N_\theta}-(r+1)\frac{\partial_\alpha h}{h}\overset{r}{\psi}_{N_\theta}-\sum_{s=r+1}^{N_\theta+2}\frac{\overset{s}{\psi}_{N_\theta}}{h}\left(s+\frac{1}{2}\right)\left(\partial_\alpha h^{+}-(-1)^{r+s}\partial_\alpha h^{-}\right)\right)\times$$

$$\left.\times\left(\partial_\alpha \overset{r}{\varphi}_{N_\theta}-(r+1)\frac{\partial_\alpha h}{h}\overset{r}{\varphi}_{N_\theta}-\sum_{\hat{s}=r+1}^{N_\theta+2}\frac{\overset{\hat{s}}{\varphi}_{N_\theta}}{h}\left(\hat{s}+\frac{1}{2}\right)\left(\partial_\alpha h^{+}-(-1)^{r+\hat{s}}\partial_\alpha h^{-}\right)\right)\right]d\omega,$$

$$b_{\mathbf{N}N_\theta}(\vec{\varphi}_{N_\theta},\vec{v}_{\mathbf{N}}) = -\frac{\chi\eta\rho}{\gamma}b^{\theta}_{\mathbf{N}N_\theta}(\vec{v}_{\mathbf{N}},\vec{\varphi}_{N_\theta}) = \frac{\gamma}{\rho}\sum_{r=0}^{N_3}\left(r+\frac{1}{2}\right)\int_{\omega}\frac{1}{h^2}\left(\sum_{s=r}^{N_\theta+2}(s+\frac{1}{2})\overset{s}{\varphi}_{N_\theta}(1-(-1)^{r+s})\right)\overset{r}{v}_{\mathbf{N}3}\,d\omega+$$

$$+\frac{\gamma}{\rho}\sum_{\alpha=1}^{2}\sum_{r=0}^{N_\alpha}\left(r+\frac{1}{2}\right)\int_{\omega}\frac{1}{h}\left(\partial_\alpha \overset{r}{\varphi}_{N_\theta}-(r+1)\frac{\partial_\alpha h}{h}\overset{r}{\varphi}_{N_\theta}-\sum_{s=r+1}^{N_\theta+2}\frac{\overset{s}{\varphi}_{N_\theta}}{h}\left(s+\frac{1}{2}\right)\left(\partial_\alpha h^{+}-(-1)^{r+s}\partial_\alpha h^{-}\right)\right)\overset{r}{v}_{\mathbf{N}\alpha}\,d\omega,$$

where $N_{\max}=\max\{N_1,N_2,N_3\}$, $\overset{r}{e}_{ij}(\vec{v}_{\mathbf{N}})=\frac{1}{2}\left(\partial_i(\overset{r}{v}_{\mathbf{N}j})+\partial_j(\overset{r}{v}_{\mathbf{N}i})+\overset{r}{\tilde{e}}_{ij}(\vec{v}_{\mathbf{N}})\right)$,

$$\overset{r}{\tilde{e}}_{ij}(\vec{v}_{\mathbf{N}})=-\frac{r+1}{h}\left(\partial_i h\overset{r}{v}_{\mathbf{N}j}+\partial_j h\overset{r}{v}_{\mathbf{N}i}\right)-\sum_{s=r+1}^{N_{\max}}\frac{1}{h}\left(s+\frac{1}{2}\right)\left(\overset{s}{v}_{\mathbf{N}j}(\partial_i h^{+}-(-1)^{r+s}\partial_i h^{-})+\right.$$

$$\left.+\overset{s}{v}_{\mathbf{N}i}(\partial_j h^{+}-(-1)^{r+s}\partial_j h^{-})\right)+\sum_{s=r}^{N_{\max}}\frac{1}{h}\left(s+\frac{1}{2}\right)(1-(-1)^{r+s})\left(\frac{(i-1)(i-2)}{2}\overset{s}{v}_{\mathbf{N}j}+\frac{(j-1)(j-2)}{2}\overset{s}{v}_{\mathbf{N}i}\right),$$

and $\overset{N_\theta+1}{\psi}_{N_\theta}$, $\overset{N_\theta+2}{\psi}_{N_\theta}$ are defined in the same way as $\overset{N_\theta+1}{\varphi}_{N_\theta}$, $\overset{N_\theta+2}{\varphi}_{N_\theta}$. The linear forms $L_{\mathbf{N}}$, $L^{\theta}_{N_\theta}$ are defined by the right-hand sides of the equations (5), (6) and are given by

$$L_{\mathbf{N}}(\vec{v}_{\mathbf{N}})=\sum_{i=1}^{3}\sum_{r_i=0}^{N_i}\left(r_i+\frac{1}{2}\right)\left[\int_{\omega}\frac{1}{h}\overset{r_i}{v}_{\mathbf{N}i}\left(\overset{r_i}{f}_i+\frac{1}{\rho}g_i^{+}\lambda_{+}+\frac{1}{\rho}g_i^{-}\lambda_{-}(-1)^{r_i}\right)d\omega+\frac{1}{\rho}\int_{\gamma_1}\frac{1}{h}\overset{r_i}{v}_{\mathbf{N}i}\overset{r_i}{g}_i\,d\gamma_1\right],$$

$$L^{\theta}_{N_\theta}(\vec{\varphi}_{N_\theta})=\sum_{r=0}^{N_\theta}\left(r+\frac{1}{2}\right)\int_{\omega}\frac{1}{h}\overset{r}{\varphi}_{N_\theta}\left(\overset{r}{f}^{\theta}-\sum_{\alpha=1}^{2}\frac{2(\alpha-1)-(-1)^{\alpha}}{2}((-1)^{r+N_\theta+\alpha}+1)\overset{N_\theta+\alpha}{f}^{\theta}\right)d\omega,$$

where $\gamma_1=\bar{\gamma}\setminus\bar{\gamma}_0$, $\lambda_{\pm}=\sqrt{1+(\partial_1 h^{\pm})^2+(\partial_2 h^{\pm})^2}$, $\overset{r}{\varphi}=\int_{h^-}^{h^+}\varphi P_r(z)dx_3$, for all functions $\varphi\in L^2(\Omega)$, $r\in\mathbf{N}\cup\{0\}$, g_i^{+} and g_i^{-} are restrictions of g_i on the upper Γ^{+} and the lower Γ^{-} faces of the shell.

For the constructed two-dimensional initial-boundary value problems (10)-(12) the following existence and uniqueness theorem is proved.

Theorem 1. *If* ω *and functions* h^{+}, h^{-} *are such that* Ω *is a Lipschitz domain,* $\mu>0$, $3\lambda+2\mu>0$, $\chi>0$, $\gamma/\eta>0$, $\vec{w}_{\mathbf{N}0}\in\vec{V}_{\mathbf{N}}(\omega)$, $\vec{w}_{\mathbf{N}1}\in\vec{H}_{\mathbf{N}}(\omega)$, $\vec{\zeta}_{N_\theta 0}\in\vec{H}^{\theta}_{N_\theta}(\omega)$, *and the functions* $\overset{r_i}{f}_i$, $\overset{r_i}{g}_i$, $g_i^{\pm}$ $(r_i=\overline{0,N_i}, i=\overline{1,3})$, $\overset{r}{f}^{\theta}$ $(r=\overline{0,N_\theta+2})$ *satisfy the following conditions*

$$h^{-1/6}\overset{r_i}{f_i}\in L^2(0,T;L^{6/5}(\omega)),\ \ \lambda_{\pm}^{3/4}g_i^{\pm}\in L^2(0,T;L^{4/3}(\omega)),$$

$$h^{-1/4}\overset{r_i}{g_i}\in L^2(0,T;L^{4/3}(\gamma_1)),\quad r_i=\overline{0,N_i},\ i=\overline{1,3},$$

$$h^{-1/6}\left(\overset{r}{f^{\theta}}-\frac{1-(-1)^{r+N_\theta}}{2}\overset{N_\theta+1}{f^{\theta}}-\frac{1+(-1)^{r+N_\theta}}{2}\overset{N_\theta+2}{f^{\theta}}\right)\in L^2(0,T;L^{6/5}(\omega)),\ \ r=\overline{0,N_\theta},$$

then the dynamical two-dimensional problem (10)-(12) *possesses a unique solution.*

Along with investigation of the reduced two-dimensional problem it is very important to study the relation of the constructed hierarchical model to the original three-dimensional problem. In order to formulate the corresponding theorem let us define the following anisotropic weighted Sobolev space

$$H^{1,1,s}_{h^{\pm}}(\Omega)=\{v;\partial_3^{r-1}v\in H^1(\Omega),\partial_\alpha h^{\pm}\partial_3^r v\in L^2(\Omega),\alpha=1,2,r=\overline{1,s}\},\qquad s\in\mathbf{N},$$

which is a Hilbert space equipped with the norm

$$\|v\|^2_{H^{1,1,s}_{h^{\pm}}(\Omega)}=\sum_{r=1}^{s}\left(\left\|\partial_3^{r-1}v\right\|_{H^1(\Omega)}+\sum_{\alpha=1}^{2}\left(\left\|\partial_\alpha h^{+}\partial_3^r v\right\|_{L^2(\Omega)}+\left\|\partial_\alpha h^{-}\partial_3^r v\right\|_{L^2(\Omega)}\right)\right),\qquad s\in\mathbf{N}.$$

Theorem 2. *If* $\mathbf{u}_0\in\mathbf{V}(\Omega)$, $\mathbf{u}_1\in\mathbf{L}^2(\Omega)$, $\theta_0\in L^2(\Omega)$, $\mathbf{f}=(f_i)_{i=1}^3\in L^2(0,T;\mathbf{L}^2(\Omega))$, $\mathbf{g},\mathbf{g}'\in L^2(0,T;\ \mathbf{L}^{4/3}(\Gamma_1))$, $f^{\theta}\in L^2(0,T;L^{6/5}(\Omega))$, $\mu>0$, $3\lambda+2\mu>0$, $\chi>0$, $\gamma/\eta>0$ *and the functions* $\mathbf{w}_{\mathbf{N}0}\in\mathbf{V}_{\mathbf{N}}(\Omega)$, $\mathbf{w}_{\mathbf{N}1}\in\mathbf{H}_{\mathbf{N}}(\Omega)$, $\zeta_{N_\theta 0}\in H^{\theta}_{N_\theta}(\Omega)$, *corresponding to the initial conditions* $\vec{w}_{\mathbf{N}0}\in\vec{V}_{\mathbf{N}}(\omega)$, $\vec{w}_{\mathbf{N}1}\in\vec{H}_{\mathbf{N}}(\omega)$, $\vec{\zeta}_{N_\theta 0}\in\vec{H}^{\theta}_{N_\theta}(\omega)$ *of two-dimensional problems, tend to* $\mathbf{u}_0$, $\mathbf{u}_1$ *and* θ_0 *in the spaces* $\mathbf{H}^1(\Omega)$, $\mathbf{L}^2(\Omega)$ *and* $L^2(\Omega)$, *respectively, as* $N_{\min}=\min\limits_{1\le i\le 3}\{N_i,N_\theta\}\to\infty$, *then the vector-function* $\mathbf{w}_{\mathbf{N}}(t)$ *and function* $\zeta_{N_\theta}(t)$ *restored from the solution* $\vec{w}_{\mathbf{N}}$ *and* $\vec{\zeta}_{N_\theta}$ *of the reduced two-dimensional problem (10)-(12), tends to the solution of the original three-dimensional problem (7)-(9),*

$$\begin{aligned}&\mathbf{w}_{\mathbf{N}}(t)\to\mathbf{u}(t) && \text{in } \mathbf{H}^1(\Omega),\\ &\mathbf{w}'_{\mathbf{N}}(t)\to\mathbf{u}'(t) && \text{in } \mathbf{L}^2(\Omega), \qquad \text{for all } t\in[0,T],\ \text{as}\ \ N_{\min}\to\infty.\\ &\zeta_{N_\theta}(t)\to\theta(t) && \text{in } L^2(\Omega),\end{aligned}$$

In addition, if $d^r\mathbf{u}/dt^r\in L^2(0,T;(H^{1,1,s_r}_{h^{\pm}}(\Omega))^3)$, $s_r\in\mathbf{N}$, $r=\overline{0,2}$, $s_0\ge s_1\ge s_2\ge 1$, $s_1\ge 2$, $\theta\in L^2(0,T;H^{1,1,s_0^{\theta}}_{h^{\pm}}(\Omega))$, $\theta'\in L^2(0,T;H^{1,1,s_1^{\theta}}_{h^{\pm}}(\Omega))$, $s_0^{\theta}\ge s_1^{\theta}\ge 1$, $s_0^{\theta}\ge 2$, *then for suitable initial data* $\vec{w}_{\mathbf{N}0}$, $\vec{w}_{\mathbf{N}1}$ *and* $\vec{\zeta}_{N_\theta 0}$ *the following estimate is valid*

$$\|\mathbf{u}'-\mathbf{w}'_{\mathbf{N}}\|_{C^0([0,T];\mathbf{L}^2(\Omega))}+\|\mathbf{u}-\mathbf{w}_{\mathbf{N}}\|_{C^0([0,T];\mathbf{H}^1(\Omega))}+$$
$$+\left\|\theta-\zeta_{N_\theta}\right\|_{C^0([0,T];L^2(\Omega))}+\left\|\theta-\zeta_{N_\theta}\right\|_{L^2(0,T;H^1(\Omega))}\le\frac{1}{(N_{\min})^s}o(T,\Omega,\widetilde{\Gamma}_0,h^{\pm},\mathbf{N},N_\theta),$$

where $s=\min\{s_2,s_1-1,s_1^{\theta},s_0^{\theta}-1\}$, $o(T,\Omega,\widetilde{\Gamma}_0,h^{\pm},\mathbf{N},N_\theta)\to 0$, *as* $N_{\min}\to\infty$.

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თერმოდრეკადი პრიზმული გარსების იერარქიული მოდელების აგება და გამოკვლევა

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