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Mathematics

On Multiple Integrals of New Type from Many-Valued Functions of Abstract Set

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ABSTRACT. In this paper we introduce the multiple integral of new types for many-valued function of abstract set. Fundamental relations between upper and lower double integrals and iterated upper and lower integrals are established, for which the generalized Fubini's theorem is obtained for both set functions and point functions. *© 2008 Bull. Georg. Natl. Acad. Sci.*

Key words: *from the right (from the left) generalized-netting partition, netting double integral, from the right (from the left) generalized-netting double integral.*

The class of sets, contained together with any two sets of their intersection, is called multiplicative and is designated by $\mathfrak{M}$.

Let $\mathfrak{M}$ be any multiplicative class of sets and $E \in \mathfrak{M}$. A finite or countable class of pairwise disjoint sets $\{E_1, E_2, \ldots\}$, belonging to class $\mathfrak{M}$ and whose union is equal to set E is called the partition of set E and is designated by DE where the sets E_k (k=1,2,...) are called the components of partition DE. If the partition contains only a finite number of components is called finite and is designated by D^*E.

Partition D_1E of set $E \in \mathfrak{M}$ is called a continuation of partition DE of set E in designation $DE \prec D_1E$, if each component of partition D_1E is a subset of some component of partition DE or, which is the same, if partition D_1E contains a partition of each component of partition DE.

Let $D_1E = \{E_1', E_2', \ldots\}$ and $D_2E = \{E_1'', E_2'', \ldots\}$ be two partitions of set $E \in \mathfrak{M}$. The product of partitions D_1E and D_2E is a partition whose components are all possible intersections $E_i' \cap E_k'' \left(i, k = 1, 2, \ldots\right)$ and is designated by $\left(D_1 \cdot D_2\right)E$. Obviously, the product of two partitions is a continuation of both partitions.

Let us designate the set of all partitions of set $E \in \mathfrak{M}$ by $\mathfrak{A}E$, and the set of all components of all possible partitions of set E by $\sum_{\mathfrak{M}}(E)$.

The class of sets contained together with any two sets of a partition of their intersection, is called generalized-multiplicative and is designated by $\mathfrak{M}'$.

It is obvious that the multiplicative class is also generalized-multiplicative. We shall give an example of a generalized-multiplicative class which is not multiplicative. Let E be any set whose power is bigger or equal to countable power and P(E) be a class of all its subsets. We take the finite subset $e \subset E$ and we shall remove the set e and all its subsets, except single-element subsets from class P(E). We shall designate the remaining class by $\mathfrak{M}'$ and we shall show that it is generalized-multiplicative, but not multiplicative. Indeed, we take set E–e and we shall present it in the form $E - e = F_1 \cup F_2$, where $F_1 \cap F_2 = \varnothing$ ($\varnothing$ - empty set). We generate sets $E_1 = F_1 \cup e$ and $E_2 = F_2 \cup e$. Then $E_1 \cap E_2 = (F_1 \cup e) \cap (F_2 \cup e) = e$. Hence, $E_1 \cap E_2 = e \notin \mathfrak{M}$. However, $\mathfrak{M}$ contains partition of set $E_1 \cap E_2 = e$.

We shall call a class of sets normal and designate it by $\mathfrak{N}$, if for every set $E \in \mathfrak{N}$ the class $\mathfrak{N}E$ of all partitions of set E is the directed relation of continuation $\succ$.

Theorem 1. *The class of sets* $\mathfrak{N}$ *is a normal class if and only if when for any set* $E \in \mathfrak{N}$ *the set* $\sum_{\mathfrak{A}}(E)$ *of all components of all possible partitions of sets* E *is a generalized-multiplicative class.*

A generalized-multiplicative class $\mathfrak{M}$ is called a semiring and is designated by P, if it contains an empty set $\varnothing$ and from $E_1, E \in P$ and $E_1 \subset E$ it follows that P contains a partition of set E whose component is set E_1.

Let X and Y be any sets. The set of all ordered pairs (x,y), where $x \in X$ and $y \in Y$, is called the Cartesian product of sets X and Y and is designated by $X \times Y$. If $A \subset X$ and $B \subset Y$, then set $E = A \times B$, contained in $X \times Y$, is called a rectangle, and sets A and B - the sides of this rectangle.

Let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be normal classes. The class of all rectangles $A \times B$, where $A \in \mathfrak{N}_1$ and $B \in \mathfrak{N}_2$ is called product of normal classes $\mathfrak{N}_1$ and $\mathfrak{N}_2$ and we shall designate it by $\mathfrak{N}_1 \otimes \mathfrak{N}_2$.

It is directly checked that the product of normal classes is also a normal class.

Proposition 1. If P_1 and P_2 are semirings, then their product $P = P_1 \otimes P_2$ is also a semiring (see [1]).

Let us introduce the following definition:

Partition $D(A \times B)$ of rectangle $A \times B \in \mathfrak{N}$ is called a netting partition, if it has the form $\{A_i \times B_k\}(i,k = 1,2,\ldots)$, where $\{A_i\}(i = 1,2,\ldots)$ is partition of side A, and $\{B_k\}(k = 1,2,\ldots)$ is partition of side B.

Partition $D(A \times B)$ of rectangle $A \times B \in \mathfrak{N}$ is called from the right generalized-netting partition, if it has a form $\{A_i \times B^i_k\}(i,k = 1,2,\ldots)$, where $\{A_i\}(i = 1,2,\ldots)$ is partition of side A and $\{B^i_k\}(k = 1,2,\ldots)$ is partition of side B corresponding to component $\{A_i\}$.

Partition $D(A \times B)$ of rectangle $A \times B \in \mathfrak{N}$ is called from the left generalized-netting partition, if it has a form $\{A^k_i \times B_k\}(k,i = 1,2,\ldots)$, where $\{B_k\}(k = 1,2,\ldots)$, is partition of side B and $\{A^k_i\}(i = 1,2,\ldots)$ is partition of party A corresponding to component $\{B_k\}$.

Partition $D(A \times B)$ of rectangle $A \times B \in \mathfrak{N}$ is called twice-netting partition if it has a form $\{A^k_i \times B^l_j\}(i,j,k,l = 1,2,\ldots)$, where $\{A_i \times B_j\}(i,j = 1,2,\ldots)$ is netting partition of rectangle $A \times B$, and $\{A^k_i \times B^l_j\}(k,l = 1,2,\ldots)$ is netting partition of rectangle $A_i \times B_j$.

It is directly checked that the set $\mathfrak{N}_n$ of all netting partitions of rectangle $A \times B \in \mathfrak{N}$ the set $\mathfrak{N}_r$ of all from the right generalized-netting partitions, the set $\mathfrak{N}_l$ of all from the left generalized-netting partitions and the set $\mathfrak{N}^2_n$ of all twice-netting partitions are directed by the relation of continuation.

Therefore, their mappings to the set of real numbers are directions and the generalized theory of limits is applicable to them (see [2], ch.II).

Proposition 2. Partition $D(A \times B)$ of rectangle $A \times B$ is a twice-netting partition if and only if when it is a product of from the left and from the right generalized-netting partitions of rectangle $A \times B$.

Proposition 3. If $A_1, \ldots, A_m$ are arbitrary finite class of sets from semiring P, then their union is presented as a class

$$A_1 \cup \ldots \cup A_m = A_1^1 \cup \ldots \cup A_1^{k_1} \cup \ldots \cup A_m^1 \cup \ldots \cup A_m^{k_m},$$

where $A_i^1, \ldots, A_i^{k_i}$ contained in set $A_i\,(i=1,\ldots,m)$ and all sets on the right belong to semiring P and are pairwise disjoint.

Proof see [1].

This proposition is not true for countable classes of sets as it is marked by mistake in [3], p.132. A counterexample is the countable class $\left\{\left[\frac{m-1}{2^n},\frac{m}{2^n}\right)\right\}\left(n=1,2,\ldots;m=1,2,\ldots,2^n\right)$ in semiring P of all semisegments $[a,b)$ from semisegment [0,1).

Proposition 4. Let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be normal classes of sets, $\mathfrak{N}_1 \otimes \mathfrak{N}_2$ - their product, rectangle $A \times B \in \mathfrak{N}_1 \otimes \mathfrak{N}_2$ and $D(A\times B)=\{A_k \times B_k\}(k=1,2,\ldots)$ partition of rectangle $A\times B$. In order for netting continuation of partition $D(A\times B)$ to exist, it is necessary and sufficient to find such partition $\{\tilde{A}_1, \tilde{A}_2, \ldots\} \subset \mathfrak{N}_1$ of set A and such partition $\{\tilde{B}_1, \tilde{B}_2, \ldots\} \subset \mathfrak{N}_2$ of set B that every $A_k\,(k=1,2,\ldots)$ respectively $B_k\,(k=1,2,\ldots)$ is union finite or counting numbers of sets from $\{\tilde{A}_1, \tilde{A}_2, \ldots\}$ respectively from $\{\tilde{B}_1, \tilde{B}_2, \ldots\}$.

Corollary 1. *Let* P_1 *and* P_2 *be semirings and* $P_1 \otimes P_2$ *is their product. Then for any finite partition* $\{A_k \times B_k\}(k=1,\ldots,n)$ *of rectangle* $A\times B \in P_1 \otimes P_2$ *there is its netting continuation.*

The corollary 1 does not extend to counting partitions. Indeed, let P_1 be a semiring of all semisegments $[a,b)$ contained in [0,1) and P_2 be a semiring of all semisegments $[a,b)$ contained in $[0,+\infty)$ and $P = P_1 \otimes P_2$ their product. We shall consider a countable partition

$$\left\{\left[\frac{m-1}{2^n},\frac{m}{2^n}\right]\times[n-1,n)\right\}\left(n=1,2,\ldots;m=1,\ldots,2^n\right)$$

of rectangle $[0,1)\times[0,+\infty)$. It is from the right generalized-netting partition, however, for it there does not exist a netting continuation.

It is said that a many-valued function μ of set E is given on the class $\mathfrak{N}$ if a set $\mu(E)$ of real numbers is assigned to each set $E \in \mathfrak{N}$ and $\mu(\varnothing)=0$.

Let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be normal classes of sets, $\mathfrak{N} = \mathfrak{N}_1 \otimes \mathfrak{N}_2$ - their product, and any many-valued function of rectangle μ is given on rectangle $A\times B \in \mathfrak{N}$.

We call the real number I netting double integral of μ on rectangle $A\times B$ and we shall denote it by the symbol

$$(\mathfrak{N})\iint_{A\times B}\mu(dA,dB),$$

if for any positive number $\varepsilon>0$ there is such netting partition $D_\varepsilon(A\times B)$ of rectangle $A\times B$ that for its any continuation $\{A_i \times B_j\}(i,j=1,2\ldots)$ at any choice value $\mu(A_i,B_j)$ $(i,j=1,2\ldots)$ the inequality

$$\left|I-\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mu(A_i,B_j)\right|<\varepsilon$$

takes place.

We call the real number I from the right (from the left) generalized-netting double integral of μ on rectangle $A\times B$ and we shall denote it by the symbol

$$(\mathfrak{N}^r)\iint_{A\times B}\mu(dA,dB),\quad \left((\mathfrak{N}^l)\iint_{A\times B}\mu(dA,dB)\right)$$

if for any positive number $\varepsilon > 0$ there is such from the right (from the left) generalized-netting partition $D_\varepsilon(A \times B)$ of rectangle $A \times B$ that for any its from the right (from the left) generalized-netting partition continuation $\{A_i \times B_j^i\}(i,j=1,2\ldots)(\{A_j^i \times B_i\}(i,j=1,2\ldots))$ and any choice value $\mu(A_i, B_j^i)(i,j=1,2\ldots)(\mu(A_j^i, B_i)(i,j=1,2\ldots))$ the inequality

$$\left| I - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mu(A_i, B_j^i) \right| < \varepsilon \quad \left(\left| I - \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \mu(A_j^i, B_i) \right| < \varepsilon \right)$$

takes place.

The definitions of the upper, lower and iterated upper and lower integral of Kolmogorroff's are given in [4].

Theorem 2. *Let* $\mathfrak{N}_1$ *and* $\mathfrak{N}_2$ *be normal classes of sets,* $\mathfrak{N} = \mathfrak{N}_1 \otimes \mathfrak{N}_2$ *- their product, and any many-valued function of rectangle* μ *be given on the rectangle* $A \times B \in \mathfrak{N}$. *Then the inequalities*

$$\begin{aligned} (\mathfrak{N}^r)\underline{\iint_{A\times B}} \mu(dA, dB) &\le (\mathfrak{N}_1)\underline{\int_{A}}\left((\mathfrak{N}_2)\underline{\int_{B}} \mu(dA, dB) \right) \le \\ &\le (\mathfrak{N}_1)\overline{\int_{A}}\left((\mathfrak{N}_2)\overline{\int_{B}} \mu(dA, dB) \right) \le (\mathfrak{N}^r)\overline{\iint_{A\times B}} \mu(dA, dB) \end{aligned} \tag{1}$$

$$\begin{aligned} (\mathfrak{N}^l)\underline{\iint_{A\times B}} \mu(dA, dB) &\le (\mathfrak{N}_1)\underline{\int_{B}}\left((\mathfrak{N}_2)\underline{\int_{A}} \mu(dA, dB) \right) \le \\ &\le (\mathfrak{N}_1)\overline{\int_{B}}\left((\mathfrak{N}_2)\overline{\int_{A}} \mu(dA, dB) \right) \le (\mathfrak{N}^l)\overline{\iint_{A\times B}} \mu(dA, dB) \end{aligned}$$

take place.

As the following inequalities take place

$$\begin{aligned} (\mathfrak{N})\underline{\iint_{A\times B}} \mu(dA, dB) &\le (\mathfrak{N}^r)\underline{\iint_{A\times B}} \mu(dA, dB) \le \\ &\le (\mathfrak{N}^r)\overline{\iint_{A\times B}} \mu(dA, dB) \le (\mathfrak{N})\overline{\iint_{A\times B}} \mu(dA, dB) \\ (\mathfrak{N})\underline{\iint_{A\times B}} \mu(dA, dB) &\le (\mathfrak{N}^l)\underline{\iint_{A\times B}} \mu(dA, dB) \le \\ &\le (\mathfrak{N}^l)\overline{\iint_{A\times B}} \mu(dA, dB) \le (\mathfrak{N})\overline{\iint_{A\times B}} \mu(dA, dB) \end{aligned}$$

from the proved theorem it follows:

Corollary 2. *Let* $\mathfrak{N}_1$ *and* $\mathfrak{N}_2$ *be normal classes of sets,* $\mathfrak{N} = \mathfrak{N}_1 \otimes \mathfrak{N}_2$ *- their product, and any many-valued function of rectangle* μ *be given on the rectangle* $A \times B \in \mathfrak{N}$. *Then the inequalities*

$$\begin{aligned} (\mathfrak{N})\underline{\iint_{A\times B}} \mu(dA, dB) &\le (\mathfrak{N}_1)\underline{\int_{A}}\left((\mathfrak{N}_2)\underline{\int_{B}} \mu(dA, dB) \right) \le \\ &\le (\mathfrak{N}_1)\overline{\int_{A}}\left((\mathfrak{N}_2)\overline{\int_{B}} \mu(dA, dB) \right) \le (\mathfrak{N})\overline{\iint_{A\times B}} \mu(dA, dB), \\ (\mathfrak{N})\underline{\iint_{A\times B}} \mu(dA, dB) &\le (\mathfrak{N}_2)\underline{\int_{B}}\left((\mathfrak{N}_1)\underline{\int_{A}} \mu(dA, dB) \right) \le \\ &\le (\mathfrak{N}_2)\overline{\int_{B}}\left((\mathfrak{N}_1)\overline{\int_{A}} \mu(dA, dB) \right) \le (\mathfrak{N})\overline{\iint_{A\times B}} \mu(dA, dB). \end{aligned}$$

take place.

Theorem 3. *Let* $\mathfrak{N}_1, \ldots, \mathfrak{N}_n$ *be normal classes of sets,* $\mathfrak{N} = \mathfrak{N}_1 \otimes \cdots \otimes \mathfrak{N}_n$ *- their product, and any many-valued function of rectangle* μ *be given on rectangle* $A_1 \times \cdots \times A_n \in \mathfrak{N}$. *Then the* $n!$ *inequalities*

$$(\mathfrak{N}) \underline{\iint}_{A_1\times\cdots\times A_n} \mu(dA_1,\ldots,dA_n) \le (\mathfrak{N}_n)\underline{\int}_{A_n}\left(\ldots\left((\mathfrak{N}_1)\underline{\int}_{A_1}\mu(dA_1,\ldots,dA_n)\right)\ldots\right)\le$$

$$\le (\mathfrak{N}_n)\overline{\int}_{A_n}\left(\ldots\left((\mathfrak{N}_1)\overline{\int}_{A_1}\mu(dA_1,\ldots,dA_n)\right)\ldots\right)\le (\mathfrak{N}) \overline{\iint}_{A_1\times\cdots\times A_n} \mu(dA_1,\ldots,dA_n),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(\mathfrak{N}) \underline{\iint}_{A_n\times\cdots\times A_1} \mu(dA_1,\ldots,dA_n) \le (\mathfrak{N}_1)\underline{\int}_{A_1}\left(\ldots\left((\mathfrak{N}_n)\underline{\int}_{A_n}\mu(dA_1,\ldots,dA_n)\right)\ldots\right)\le$$

$$\le (\mathfrak{N}_1)\overline{\int}_{A_1}\left(\ldots\left((\mathfrak{N}_n)\overline{\int}_{A_n}\mu(dA_1,\ldots,dA_n)\right)\ldots\right)\le (\mathfrak{N}) \overline{\iint}_{A_n\times\cdots\times A_1} \mu(dA_1,\ldots,dA_n)$$

take place.

The proof is directly obtained from corollary 2 by application of the method of mathematical induction.

მათემატიკა

აბსტრაქტული სიმრავლის მრავალსახა ფუნქციის ახალი ტიპის ჯერადი ინტეგრალის შესახებ

დ. გოგუაძე, პ. ქარჩავა

ი. ჯავახიშვილის თბილისის სახელმწიფო უნივერსიტეტი

(წარმოდგენილია აკადემიის წევრის ვ. კოკილაშვილის მიერ)

ნაშრომში შემოღებულია აბსტრაქტული სიმრავლის მრავალსახა ფუნქციის ახალი ტიპის ჯერადი ინტეგრალები. დადგენილია ფუნდამენტური კავშირები ზედა და ქვედა და განმეორებით ზედა და ქვედა ინტეგრალებს შორის, რომელთაგანაც მიიღება ფუბინის თეორემის თავისებური განზოგადებები როგორც სიმრავლის, ასევე წერტილის ფუნქციებისათვის.

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