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# An Efficient Algorithm to Decide the Knot Problem

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**ABSTRACT. We study knot representations by sequences $\alpha$ of oriented arcs $x_1, x_2,..., x_m$, which are connected, alternating below and above the 2-sphere $S_2$ with a crossing free projection on a segment of a circle on the $S_2$, the starting point $A$ of $x_1$ is connected with the end point $B$ of $x_m$ by a crossing free string $L$ on $S_2$ oriented from $B$ to $A$. Each knot projection we represent by such a pair $(\alpha,L)$. Each such representation can be described uniquely up to isomorphisms of the 2-sphere by its signature, a finite word $\sigma_{(\alpha,L)}$ over an alphabet.**

$$X:\{x_1^{\epsilon_1}, x_2^{\epsilon_2},...,x_m^{\epsilon_m}\},\ \ \epsilon_i\in\{1,-1\}.$$

**We define transformations of the knot projections $\mathcal{K}$ on $S_2$ called normalizations or extended normalizations into arcade-string representations $(\alpha_{\mathcal{K}},L_{\mathcal{K}})$ called *AFL*.**

**These constructions define to each knot K a formal language $\mathcal{L}_{\mathrm{K}}$ defined by the set of the signatures $\sigma_{\alpha\mathcal{K}}$ of the normalizations of the projections $\mathcal{K}$ of the knot K. The equivalence of knots K and K′ can be described by the relation**

$$\mathcal{L}_{\mathrm{K}} \cap \mathcal{L}_{\mathrm{K}} \neq \varnothing .$$

**We prove that this relation is decidable in time $O\left(n^2 \cdot 2^{\frac{n}{3}}\right)$ for projections $\mathcal{K}, \mathcal{K}'$ with $n \geq$ the numbers of the crossing points of $\mathcal{K}$ and $\mathcal{K}'$. If $\mathcal{K}'$ is a circle the equivalence is decidable in time $O\left(2^{\frac{n}{3}}\right)$.** *© 2008 Bull. Georg. Natl. Acad. Sci.*

***Key words:*** *knot problem, AFL representation of knots, Reidemeister moves.*

## 1. Introduction

### 1.1. The Idea and Definitions

For an Introduction to Knot Theory see [4], [3]. The author has discussed in [5] and [6] a class of knot representations which had been suggested to him by Kurt Reidemeister [1]. These representations are based on a remark of K.F. Gauss [2], that each knot has projections on the plane which can be decomposed in two simple strings, this means strings without doppel points. Figure 1 shows as an example the projection of a trefoil knot and a decomposition of the projection in two simple segments defined by the points *A, B.* Reidemeister handled both strings in an unsymmetrical manner. One of the strings called Faden $L$ remains in the plan, the other one he moves into the $\mathbf{R}^3$ forming an arcade $\alpha$ with arcs alternating on the upper- and the lower sides of the plane. The theory becomes simpler if we use projections on the 2-*sphere* $S_2$ instead of the plain to build the arcade- string configuration on.

We may assume that the projection of the arcade forms a straight line on $S_2$ The projections of the upper arcs we will represent by red lines, the projections of the arcs under the plane by blue lines. We assume the knot to be oriented and the orientation transferred to the projection. The projections of the sequence of arcs are numbered and oriented corresponding to the orientation of the knot. The $i$-th arc of the arcade gets the name $s_i$ if it is the projection of an upper arc and the name $t_i$ in the other case. We may assume that the sequence of the projections of the arcs of the arcade alternates in its color. An arcade projection therefore can be considered as an alternating finite sequence $\alpha$

$$s_1, t_2, s_3, t_4, \ldots \quad \text{or} \quad t_1, s_2, t_3, s_4, \ldots$$

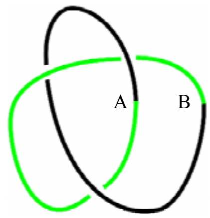



Fig. 1.

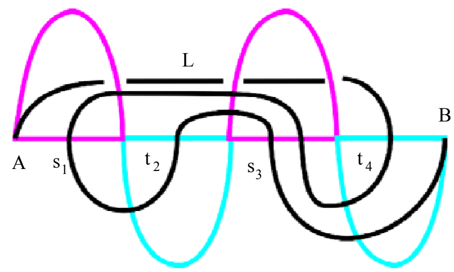



Fig. 2.

ending with a red arcade $s_n$ or a blue-end arcade $t_n$. The pairs $(\alpha, L)$ are the arcade-string-configurations (Arkaden-Faden-Lagen) introduced as knot representations by Reidemeister [5]. Figure 2 shows such a representation of the knot $4_1$ [3] p.363. For shortness we call the arcade-string-configurations *AFL*. We assign to each *AFL* $(\alpha, L)$ a signature

$$\sigma_\alpha(L) := \{x_{i_1}^{\epsilon_1} * x_{i_2}^{\epsilon_2} * \ldots * x_{i_n}^{\epsilon_n}\} \text{ for } x_{i_l} \in \{s,t\} \text{ and } \epsilon_i \in \{+1,-1\}.$$

which is defined as follows: Let

$$P_1, P_2, \ldots, P_n$$

be the sequence of the crossing-points of $L$ with the arcade $\alpha$ in the order they appear on $L$ relative to the orientation of the knot. The alphabet element $x_{i_l}$ belongs to the point $P_l$. If $P_l$ is crossing-point of $L$ with a red arc of $\alpha$, then we define $x = s$ else $x = t$ and we define $i_l := k$ if the arc is the $k$-th element in the enumeration of the arcs of $\alpha$. We define $\epsilon_l := 1$ iff $L$ crosses $x_k$ from left to right else $\epsilon_l := -1$ relative to the knot orientation. For our example Figure 2 we get relative to both of the possible orientations of the knot

$$\sigma_\alpha(L) = t_1 * s_2^{-1} * t_3.$$

The exponents of the variables are independent from the orientation of the knot. Changing the orientation results in the reflection of the signature and the numbering of the arcs. The trefoil projection in Figure 1 is an example with the same signature for both orientations. Turning the knot round the arcade as axis does not change the signature up to the trivial isomorphism, which exchanges $s$ and $t$ variables.

In [5,6] it has been shown that reductions of the signature $\sigma(L)$ by applying sequences of substitutions of the following rules (1),...,(4) generate words, which not in each case represent *AFLs* but always projections of the same knot. But it is possible to apply the reduction rules in such orders that each reduction step corresponds to an *AFL*, which can be constructed from the original *AFL* by applying a sequence of the Reidemeister moves of type $R_1$ and $R_2$. Here we use that the arcade is built on $S_2$.

$$x_{i_l}^{\epsilon} * x_{i_l}^{-\epsilon} \to 1 \text{ for } x \in \{s,t\}, \tag{1}$$

$$x_{i_1}^{r} \to 1 \text{ for } i_1 = n, \quad x_{i_n}^{r} \to 1 \text{ for } i_n = 1, \tag{2}$$

$$x_j \to x_{j-2} \text{ for } i > 0,\ j > i+1, \text{ and } x_{i+1} \text{ not in } \sigma(F), \tag{3}$$

$$i \to i-1 \text{ for } i > 1,\ x_1 \text{ not in } \sigma(F). \tag{4}$$

We use these rules to assign to each knot **K** formal languages $\mathcal{L}_{\mathbf{K}}$ over an infinite alphabet $X$.

$$S := \{s_i^{\epsilon} : i \in N, \epsilon \in \{-1,1\}\},\ T := \{t_i^{\epsilon} : i \in N, \epsilon \in \{-1,1\}\},\ \mathrm{X} := S \cup T.$$

The languages have the special structure

$$\mathcal{L} \subset (S \cdot T)^* \cup (T \cdot S)^* \cup T \cdot (S \cdot T)^* \cup (S \cdot T)^* \cdot S.$$

The operation "·" is the product, which concatenates the sequences of the free monoid $X^*$. To each knot **K** belongs an infinite set of knot projections $K$ and to each oriented projection $\mathcal{K}$ *normalizations* to *AFLs* $(\alpha_{\mathcal{K}}, L_{\mathcal{K}})$, which we describe uniquely by their signature $\sigma_{\alpha_{\mathcal{K}}}(L_{\mathcal{K}})$. We will choose special normalization algorithms and define for each oriented knot projection $\mathcal{K}$ of the not oriented knot **K** relative to the chosen class of normalizations.

$\mathcal{L}_{\mathbf{K}} := \{\sigma_\alpha(L) : (\alpha, L)$ is a normalization of an oriented projection $\mathcal{K}$ of $\mathbf{K}\}$.

Two knots **K**, **K′** are equal iff $\mathcal{L}_{\mathbf{K}} = \mathcal{L}_{\mathbf{K}'}$ holds. Equivalent to $\mathcal{L}_{\mathbf{K}} = \mathcal{L}_{\mathbf{K}'}$ is $\mathcal{L}_{\mathbf{K}} \cap \mathcal{L}_{\mathbf{K}'} \neq \emptyset$. This means that the word problem of these languages is equivalent to the knot problem.

We define two special types $v^1$ and $v^2$ of moves of points of $K$ on the arcades. The *normalizations* $v$ are sequences of such moves. We prove that to each Reidemeister-move $\mathcal{K} \to \mathcal{K}'$ there exist normalizations $v$ of $\mathcal{K}$ and the normalizations $v'$ of $\mathcal{K}'$ such that the resulting *AFLs* are isomorphic. This means that the corresponding signatures are equal. The normalization steps include reductions achieved by $R_{1-}$- and $R_{2-}$-moves corresponding to the rules (1),...,(4). The proof needs the assumption that the Reidemeister move transforming $\mathcal{K}$ into $\mathcal{K}'$ does not concern parts of the arcade. Therefore we have to consider the moves on the 2-sphere $S_2$.

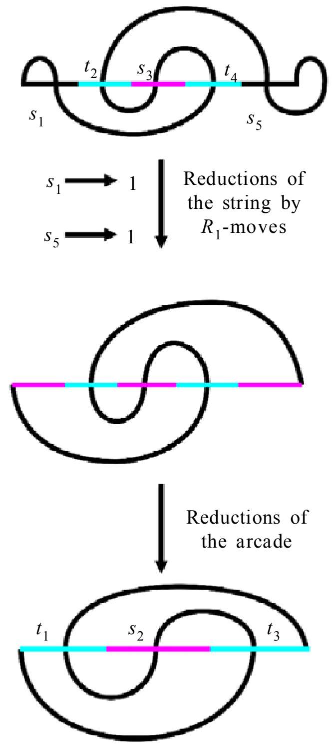



Fig. 3.

### 1.2. Examples

**Example 1.** - see Figure 3 - shows the reduction of an *AFL* representing a trefoil. We see the elementary relation between the Reidemeister moves $R_1$ and $R_2$ and the reduction rules applied to the signatures

$$\sigma_1 = s_5^{-1} \cdot t_2 \cdot s_3^{-1} \cdot t_4 \cdot s_1^{-3}, \;\; \sigma_2 = t_2 \cdot s_3^{-1} \cdot t_4, \;\; \sigma_3 = t_1 \cdot s_2^{-1} \cdot t_3.$$

**Example 2.** - Figure 4 - shows the reduction of an *AFL* representing a circle. The Reidemeister moves of type $R_1$ and $R_2$ in the graphics correspond to the following reduction steps of the related signatures.

$$\begin{aligned}
&\sigma_1 = s_1^{-1} \cdot t_2 \cdot s_3^{-1} \cdot s_3 \cdot t_4^{-1} \\
&\quad \downarrow \;\; s_3 \cdot s_3^{-1} \to 1 \\
&\sigma_2 = s_1^{-1} \cdot t_2 \cdot t_4^{-1} \\
&\quad \downarrow \;\; s_3 \to t_2, \; t_4 \to t_2 \\
&\sigma_3 = s_1^{-1} \cdot t_2 \cdot t_2^{-1} \\
&\quad \downarrow \;\; t_2 \cdot t_2^{-1} \to 1 \\
&\sigma_4 = s_1^{-1} \\
&\quad \downarrow \;\; t_2 \to s_1, \; s_1 \to 1 \\
&\sigma_5 = 1
\end{aligned}$$

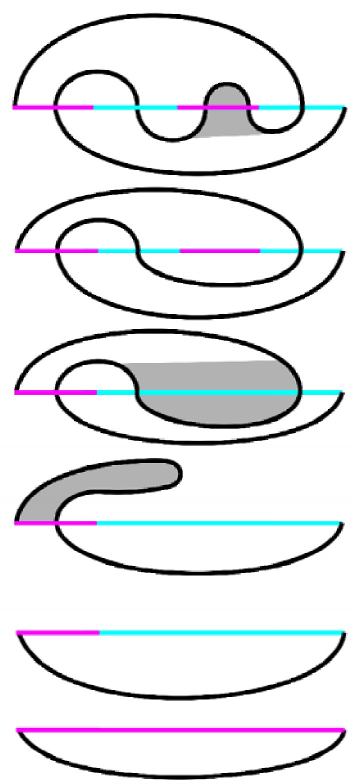


Fig. 4.

## 2. Reductions

In the first subsection of this section we prove that the result of the reduction of the signatures $\sigma_\alpha(l)$ is independent from the order of the reduction steps. In the second subsection we show that this reduction corresponds to a reduction of the *AFL* by $R_1$- and $R_2$-moves.

### 2.1. The Reduction of the Signatures

It is well known from the theory of free groups that the application of the reductions (1)

$$x_i \cdot x_i^{-1} \to 1 \text{ and } x_i^{-1} \cdot x_i \to 1$$

is commutative. In other words the result of reducing a word $\omega$ relative to these rules completely onto a word $\overline{\omega}$ is independent from the order the rules will be applied. This remains true if we add the reduction rules (2): If the first of the rules (2) can be applied to $\omega$, then $\omega$ has the form

$$\omega = x_n^{\epsilon} \cdot \widetilde{\omega} \to \widetilde{\omega}.$$

If a production of (1) can be applied on $\omega$ then it holds

$$\omega = \omega_1 \cdot x^{\eta} \cdot x^{-\eta} \cdot \omega_2.$$

If $\omega_1 \neq 1$ then we can write $\omega_1 = x_n^{\epsilon} \cdot \omega_1'$ and have as results for the two possible reductions

$$\omega = x_n^{\epsilon} \cdot \omega_1' \cdot x_i^{\eta} \cdot x_i^{-\eta} \cdot \omega_2 \to x_n^{\epsilon} \cdot \omega_1' \cdot \omega_2 \to \omega_1' \cdot \omega_2$$

$$\omega = x_n^{\epsilon} \cdot \omega_1' \cdot x_i^{\eta} \cdot x_i^{-\eta} \cdot \omega_2 \to \omega_1' \cdot x_i^{\eta} \cdot x_i^{-\eta} \cdot \omega_2 \to \omega_1' \cdot \omega_2.$$

If $\omega_1 = 1$ then the corresponding reductions have the form

$$\omega = x_n^{\epsilon} \cdot x_n^{-\epsilon} \to \omega_2,$$

$$\omega = x_n^{\epsilon} \cdot x_n^{-\epsilon} \cdot \omega_2 \to x_n^{-\epsilon} \cdot \omega_2 \to \omega_2.$$

We see that the resulting short words under reductions of type $\{(1),(2)\}$ are independent from the order we apply the reduction rules.

We now discuss the complete reduction system $\{(1),(2),(3),(4)\}$. The rules (3) and (4) identify some variables and in connection with this produce a shift in the variable names. It is clear that each production, which could be applied before an application of a rule from (3),(4) can be applied afterwards too in some cases with shifted variables. But after the application of rules from (3),(4) there may be more reductions applicable on the basis of the variable identifications. On the basis of our statement about the unique short-word under the application of the rules from (1),(2) we see that we get a uniquely determined short-word, if we before each application of a rule from (3),(4) reduce the word relative to (1),(2) completely and reduce the result of the last application of a rule from (3),(4) relative to (1),(2) completely. From this it follows, together with the observation that each reduction applicable before a variable identification can be applied after this application with a shifted rule, it follows that for each word $\omega$ there exists one and only one short-word. So we have

**Lemma 1**. *For each word $\omega \in \mathcal{L}_K$ there exists exactly one short-word $\widetilde{\omega}$. $\widetilde{\omega}$ can be constructed in linear time relative to the length $|\omega|$ of $\omega$.*

### 2.2. The Reduction of the *AFL*

Let $(\alpha, L)$ be an *AFL* and $\omega := \sigma_\alpha(L)$ the signature of $L$ relative to the arcade $\alpha$. It is obvious that each Reidemeister move $R2$ of the string $L$, which removes two crossing-points with the arc $\alpha_i$ of the arcade $\alpha$, corresponds to a reduction of the signature of the type

$$\omega = \omega_1 \cdot x_n^{\epsilon} \cdot x_n^{-\epsilon} \cdot \omega_2 \to \omega_1 \cdot \omega_2.$$

But not to each such decomposition of the signature corresponds a Reidemeister move $R_2$, which removes the related crossing-points of the *AFL*. There exist *AFLs* with the signature $\omega$ that has a decomposition

$$\omega = x_n^{\eta} \cdot \omega_1 \cdot x_i \cdot x_i^{-1} \cdot \omega_2 \cdot x_i \cdot x_i^{-1} \cdot x_1 \cdot \omega_3 \cdot x_n^{-1} \cdot x_n \cdot \omega_4$$

which corresponds to a situation as described by Figure 5.

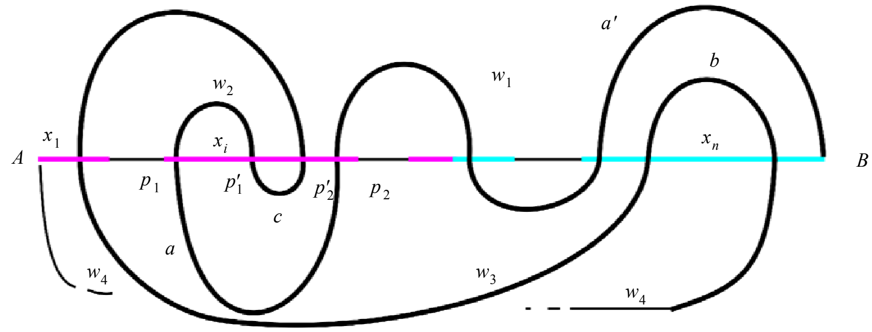



Fig. 5.

Because each *AFL* has only a finite number of crossing-points to each pair $x_i^{\epsilon} \cdot x_i^{-\epsilon}$, which appears in the signature, there exists a segment *a* of *L* that crosses the arc $\alpha_i$ in two Points $P_1$, $P_2$, which are neighbors on *L* but not necessarily on $\alpha_i$. The same difficulty concerns the segment that corresponds to the prefix $x_n^{\eta}$ of the signature. But because the string *L* and the arcade $\alpha$ are simple curves there exist in these cases other segments *b* of the string *L* with pairs $P_1', P_2'$ of crossing-points, which are neighbors on *L* and situated on the arcade between $P_1$, $P_2$. By iterating this argument we find in each case a segment *c* of *L* with crossing-points, which are neighbors on $\alpha_i$ and on *c*. This means that there exists a sequence of reductions of the signature which corresponds to a sequence of Reidemeister moves of type $R_1$ and $R_2$, which removes the corresponding crossing-points. From this follows the theorem. For a proof with more details see [6].

**Theorem 1.** *To each AFL* $(\alpha, L)$ *with the signature* $\omega := \sigma_\alpha(L)$ *there exists a AFL* $(\alpha', L')$ *and the signature* $\omega' := \sigma_{\alpha'}(L')$ *such that it holds:* $\omega \to \omega'$, $\omega'$ *is reduced and* $(\alpha, L)$ *can be transformed by a sequence of Reidemeister moves of type* $R_1$ *and* $R_2$ *and arc reductions applied on* $(\alpha, L)$ *into* $(\alpha', L')$.

## 3. Normalization

Let $\mathcal{K}$ be an oriented knot projection and *a* a segment of $\mathcal{K}$ in the same direction oriented as $\mathcal{K}$ is and without any crossing points of $\mathcal{K}$ on it. Let *A* be the start and *B* the end point of *a*. Let $P_1, P_2, \ldots, P_n$ be the set of crossing points of *K* in the order the points appear if we walk from *A* to *B* on $\mathcal{K}$ against the orientation on $\mathcal{K}$. We define the process of normalization by induction.

**Definition 1.** If *the set of crossing points is empty there is nothing to do. In the other case there exists a crossing free segment* $seg_1 := (P_1, A)$ *on* $\mathcal{K}$ *oriented as* $\mathcal{K}$ *is and a segment* $c_1$ *crossing* $seg_1$ *in* $P_1$ *which does not cross* $\mathcal{K}$ *in any other point. We move* $c_1$ *in a loop, its end points fixed with* $seg_1$ *as middle line onto such that the final loop never touches* $\mathcal{K}$ *beside in the new crossing point* $P_1'$ *in the inner part of a.*

*If* $c_1$ *is under crossing in* $P_1$ *then we color a red in the other case blue. The loop together with* $c_1$ *defines a tape* $tape_1$. *The colored segment a we define as* $\alpha_1$. *The new knot projection we call* $\mathcal{K}_1$.

*Assume the points* $P_1, \ldots, P_i$, $i < n$ *being moved onto a and it resulted in the knot projection* $\mathcal{K}_i$, *an arcade* $\alpha_i$ *on a and in the crossing free segment* $seg_{i+1}$ *oriented as* $\mathcal{K}_i$ *and connecting* $P_{i+i}$ *with A on* $\mathcal{K}_i$. $seg_{i+1}$ *may cross* $\alpha_i$ *several times.*

*We define the move* $i+1$ *as follows: We choose a small segment* $c_{i+1}$ *crossing* $seg_{i+1}$ *in* $P_{i+1}$ *and not touching* $\mathcal{K}_i$ *anywhere else. We move* $c_{i+1}$ *with fixed end points along* $seg_{i+1}$ *as middle line such that the loop does not touch* $\mathcal{K}_i$ *and itself besides of crossings of arcs of* $\alpha_{i+1}$ *and we end the move after having reached an inner point* $P_{i+1}'$ *of the first arc* $\alpha_1$ of $c_{i+1}$ *was under-crossing and the first arc has the color red, then we do not modify the arcade. In the other case we substitute* $a_1$ *by two new arcs, the first one colored blue, the second red. This we do in such a way that* $P_{i+1}'$ *lies on the blue arc and the old crossing points remain on the red part of the arc. If* $c_{i+1}$ *is over-crossing, then we proceed as before only changing the role of blue and red. Then we renumber the arcs: If we have not constructed a new arc the numbering of the arcs remains unchanged. In the other case the new arc gets the number* 1 *and the numbers of the other arcs will be increased by* 1.

*The construction is not yet defined uniquely. We have to define how the loop behaves relative to arcs of the arcade* $a_i$, *which are crossed by the moved* $c_{i+1}$. *We demand that the loop crosses always the same arc as* $seg_{i+1}$ *does and it does it on the same side: it undercrosses the arc if* $seg_{i+1}$ *does it and it overcrosses if* $seg_{i+1}$ *does it.*

*We define the new arcade as* $\alpha_{i+1}$ *and the new projection of our knot including the partial AFL as* $\mathcal{K}'_{i+1}$. *We get the* $\mathcal{K}_{i+1}$ as *the result of a complete reduction of the partial AFL of* $\mathcal{K}'_{i+1}$. *The step from* $\mathcal{K}_i$ *to* $\mathcal{K}_{i+1}$ *we call a* ***normalization step*** *of type* $v^1$. $A(\mathcal{K}, \alpha)$-***normalization*** *is the sequence of normalization steps of* $\mathcal{K}$ *relative to a, which moves each crossing point of* $\mathcal{K}$ *onto the arcade* $\alpha$ *built on a.*

**Remark 1:** in the former version [7] of this paper we considered two types $v^1$, $v^2$ of normalizations, $v^2$ differs from $v^2$ in the cases where there exist two possibilities to pass an arc of *a:* as defined or by "jumping over the arc" this means by substituting the considered arc by five arcs: the arc, which only passes the middle line, the two arcs of a different colour, which are passed by the moving loop and the two remaining arcs with the other crossings of the old arc (See Fig. 6).

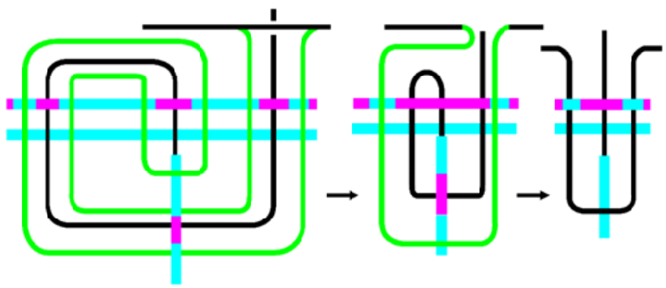


Fig. 6.

We will later introduce a special version of the normalizations of type $v^2$, which we will call *extended normalizations.*

**Remark 2:** The loops of index $i$ may be used as parts of segments $seg_j$ for $j > i$. If this happened in each normalization step, we would generate a sequence of configurations with an exponentially growing complexity. Including the reduction into each normalization step reduces this complexity explosion, as we will prove later.

The defined normalization is additionally guaranteed by the fact that the achieved *AFL* up to isomorphisms *s* is uniquely defined.

**Remark 3:** The order of normalization steps and reductions in general are not commutative. The following figure shows an example. The right side of the figure shows as the result of the normalization step of a loop generated by a $R_1$-move. We see that we may reduce the arcade after the normalization step, which moved the crossing point on the arcade. But, if we apply after the move of this point on the arcade another move on the arcade, as described by the left part of figure 7, then a reduction to the configuration described by the right side of the figure is impossible.

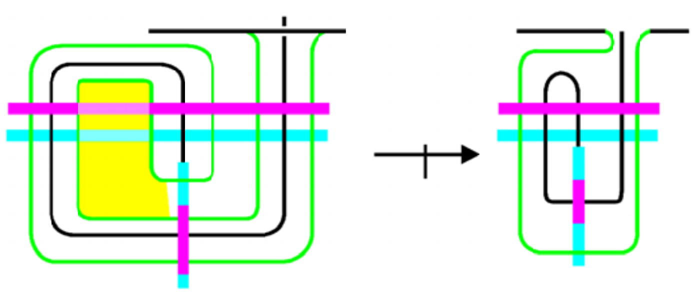


Fig. 7.

### 3.1. Reidemeister Moves on the 2-Sphere

In the following we will consider Reidemeister moves $\mathcal{K} \to \mathcal{K}'$ and we will prove invariance properties for the *AFL* $(\alpha, L)$ belonging to these knot projections. We need in these cases that the moves do not concern or touch the segment *a,* on which our arcade has been built. Considering our costructions on the $S_2$ we may assume that there exist to each pair $\mathcal{K}$, $\mathcal{K}'$ of projections of the same knot a segment *a* common to both projections and a sequence of Reidemeister moves, which move $\mathcal{K}$ into $\mathcal{K}'$, which do not contact *a.* In the next section we study how far these moves keep the corresponding *AFL's* invariant.

## 4. The $R_1$ and $R_2$ Invariance

We assume that $\mathcal{K}$, is an oriented projection of a knot **K** and that a is a crossing free segment of $\mathcal{K}$. We assume further that there exists a sequence of $R_1$ and $R_2$ moves which don't touch *a* and generate the knot projection $\mathcal{K}'$. We will prove that the *a*-normalization of $\mathcal{K}$ remains invariant under such move sequences.

### 4.1. The $R_1$ Invariance

**Lemma 2**. *Let $\mathcal{K}$ be a knot projection and* **a** *a segment of $\mathcal{K}$, free of crossing points and let $\mathcal{K}'$ be a knot projection generated by a $R_1$-move not touching* **a** *applied on $\mathcal{K}$. In this case the AFLs $(\alpha, L)$ and $(\alpha', L')$ generated by a-normalizations are isomorphic.*

**Proof.** Let $P_1$, ...$P_m$ be the crossing points of $\mathcal{K}$ between the front point *A* of *a* and the crossing point *Q* generated by the $R_1$-move. It is clear that the partial *AFLs* $\mathcal{K}_{\Updownarrow}$ and $\mathcal{K}'_{\Updownarrow}$ generated by the normalization steps concerning the crossing points $P_1$,...,$P_m$ differ only in the loop generated by the $R_1$*-move.* The normalization step applied on point *Q* on $\mathcal{K}'$ moves *Q* on the arcade. We assume that the loop to be shifted is undercrossing. If the move of $P_m$ concerned a loop which undercrosses, then the first arc of the arcade has the colour red in the other case blue. In the first case the normalization step applied on *Q* will not generate a new arc. In the second case it will generate a new arc with colour blue. We only consider the first case, which is represented by the following Figure:

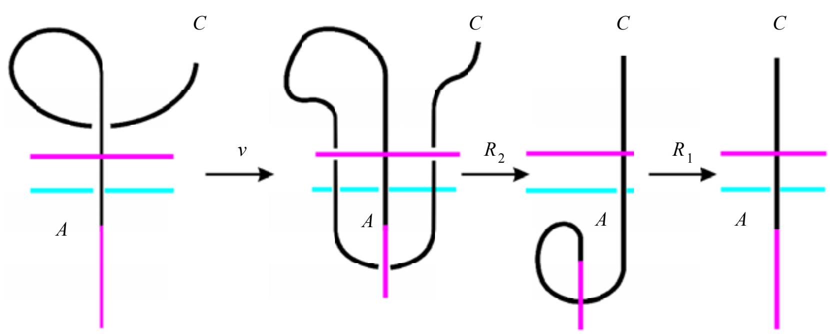



Fig. 8.

The arcs which will be crossed by the move of the normalization step are represented by a blue and a red arc. This is sufficient because the behavior relative to arcs that have crossed each other is the same. We see that the crossings of the horizontal arcs may by reduced by $R_2$-moves. with a following $R_1$-reduction we get the same configuration as before the application of the $R_1$ on $\mathcal{K}$, if the normalization of *Q* has not generated a new arc. If this is the case, we may now reduce the arcade by plotting it out. It follows that the a-normalization in both cases produces isomorphic *AFLs.*

### 4.2. The $R_2$ Invariance

We assume an analog to the case of the $R_1$ invariance, that the knot projection $\mathcal{K}$ has been moved into the projection $\mathcal{K}'$ by a $R_2$-move, which does not contact the segment *a.* Let $Q_1$, $Q_2$ be the crossing points generated by this move. Let $P_1$, ...,$P_m$ be the crossing points between *A* and $Q_1$. $Q_2$ lies between $Q_1$ and $P_{m+i}$. We assume that the new loop is undercrossing. The following figure represents this configuration before the normalization steps concerning $Q_1$ and $Q_2$ and after the normalization moves before the reduction.

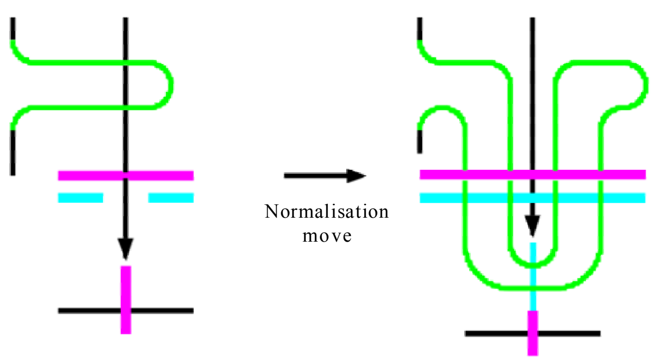



Fig. 9.

The two arcs of different colour represent all the arcs crossed by the middle line of these normalization moves. We assume in the figure that the first arc before these two normalization moves is blue. Because the moved loops are undercrossing we extend the arcade by a new red arc. We see that the reduction of the moved loop completely redraws from the arcade. In the case that the first arc is red we have to consider additionally the case that after the normalization of $Q_1$ such a reducible configuration is generated. But this does not matter, because the reduction of $x^{-1} \cdot x \cdot x^{-1}$ is associative. It follows

**Lemma 3.** *Let **a** be a segment of the knot K without any crossing points and* $(\alpha, L)$ *the arcade generated by an a-normalization of* $\mathcal{K}$. *If* $\mathcal{K}'$ *has been generated by a-normal* $R_2$*-move and* $(\alpha', L')$ *by the a-normalization of* $\mathcal{K}'$ *both AFLs are isomorphic.*

We summarize the results of both subsections as follows:

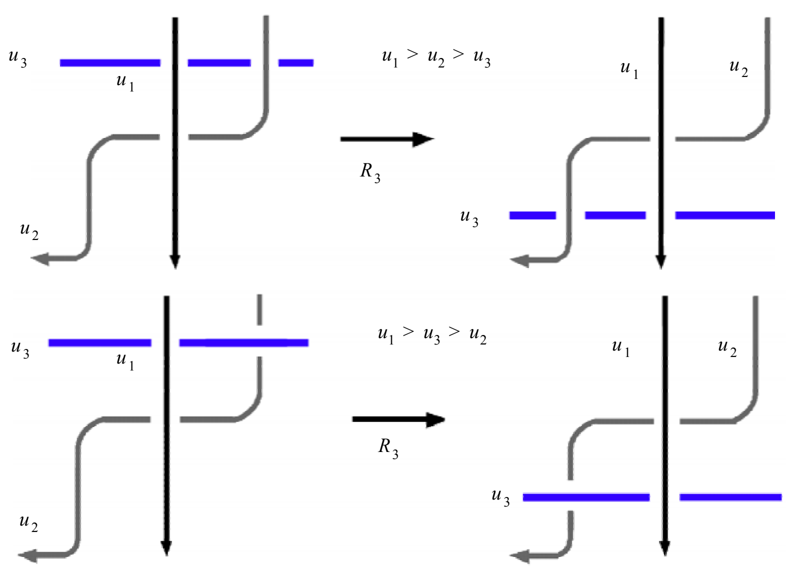
**Theorem 2.** *Let* $\mathcal{K}$, *and* $\mathcal{K}'$ *projections of the same oriented knot with a common crossing free segment **a**. If there exists a sequence of* $R_1$*- and* $R_2$*-moves, which do not contact or overjump **a**, then the a-normalizations of both projections will be isomorphic.*

## 5. The $R_3$ Invariance

We now consider the $R_3$-moves. It doesn't play any role which one of the segments we move, but the order of the segments on the oriented projection $\mathcal{K}$, plays an essential role. The following figure represents the cases we have to discuss. The order of the segments $u_3, u_2, u_1$ on $\mathcal{K}$ involved in the $R_3$-move is represented by the numbering. Their order relative to the arcade is given by the order $u_3, u_2, u_1, a$ on $\mathcal{K}$. We write $u_2 > u_1 > u_3$ if $u_2$ overcrosses $u_1$, and $u_3$ and $u_1$ overcross $u_3$. The orientation of the segments $u_1$, $u_2$ in the Figure is defined by arrows. The orientation of $u_3$ doesn't play any role, as can be easily seen in the following discussion. On the basis of the symmetry relative to the projection sphere $S_2$ we may reduce our discussions on the basis of the equivalences

$$u_1 > u_2 > u_3 \quad \cong \quad u_3 > u_2 > u_1$$
$$u_1 > u_3 > u_2 \quad \cong \quad u_2 > u_3 > u_1$$
$$u_3 > u_1 > u_2 \quad \cong \quad u_2 > u_1 > u_3$$

to the three left side relations, which are represented by the following diagrams. If we change both the orientations of $u_1$ and $u_2$, then only the left and right sides of our diagrams will be exchanged, this means that we have not to consider these cases. So it remains to discuss only the cases with an inverse orientation and of $u_2$. But these cases are symmetric of the three cases we discuss by only exchanging the left and right side of the diagrams:



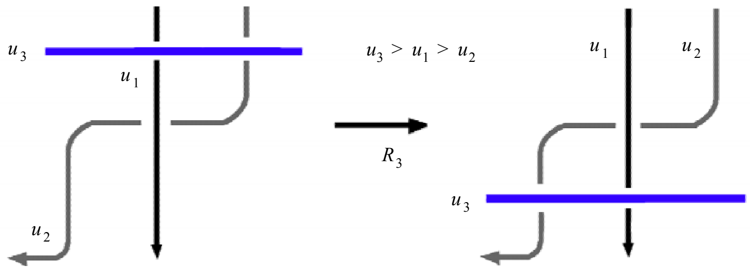



Fig. 10.

**Observation.** If we change the orientation of **K**, then the case $u_3 > u_1 > u_2$ transforms into the case $u_1 > u_3 > u_2$. This means that in these cases we achieve invariance, if we change the orientation of the knot. If we could use this property by applying first a normalization in the direction of the orientation and then in the opposite direction to get a reduced *AFL*, then it would be fine. But the application of the normalizations in opposite directions is not commutative. But we can use this property for a test on minimality of the *AFL*. If we for a given knot projection $\mathcal{K}$, and segment *a* have constructed a *AFL* $(\alpha, L)$ by a *a*-normalization, then we apply a *a*-normalization on $(\alpha, L)$ in the inverse direction and then again back. If $(\alpha, L)$ is invariant under these transformations, then it is minimal. Moreover, these *cyclic shifts* after each step represent an $a_i$-normalization, where $a_i$ runs over all possible segments for normalizations. So, if we apply this process until we get stable *AFL*'s we get all minimal representations of the knot defined by $\mathcal{K}$.

**5.1. Case $u_1 > u_2 > u_3$**

We assume that a $R_3$-move has moved the knot projection $\mathcal{K}$ into the knot projection $\mathcal{K}'$. The upper part of the following diagram represents the *Remove* of $\mathcal{K}$, into $\mathcal{K}'$. The influence of this move on the partial normalization of $\mathcal{K}$ and of $\mathcal{K}'$ is represented by the lower part of the diagram as far as it concerns the normalization moves of the three crossing points of the segments $u_1$, $u_2$, $u_3$. The normalization moves are represented in the state before the reduction. Represented are only the parts of the segments $u_2$ and $u_3$: which describe the difference of the normalization steps before and after the $R_3$-move. We see that the lower left part diagram by reduction will be transformed into the right part. This means that the reduced *AFL* $(\alpha_{\mathcal{K}}, L_{\mathcal{K}})$ is isomorphic to $(\alpha_{\mathcal{K}'}, L_{\mathcal{K}'})$. If we change the orientation of the mal segment $u_2$, then the lower part of the diagram exchanges in the essential part only the left and right sides.

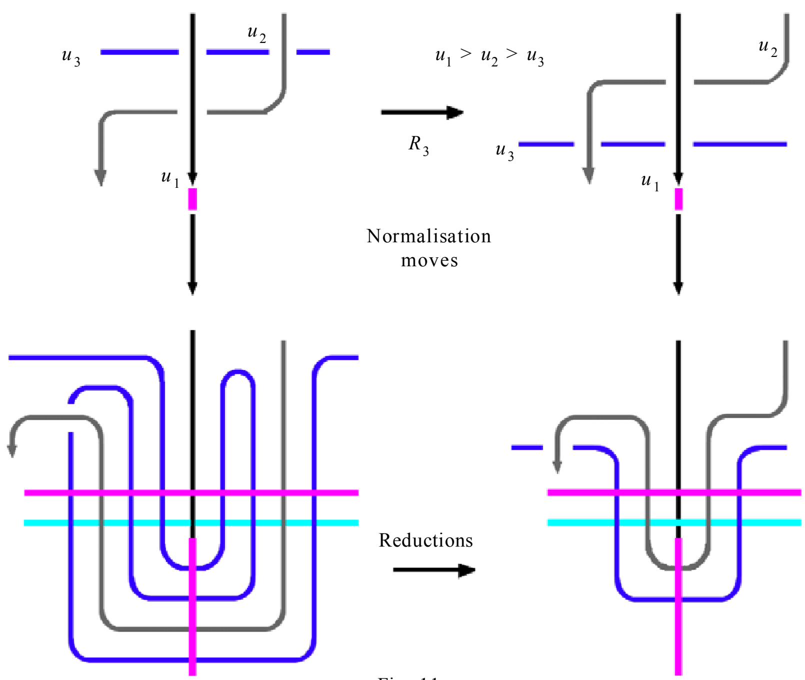



Fig. 11.

**5.2. Case $u_1 > u_3 > u_2$**

The situation is very similar to the former case and the interpretation of the following Diagram is the same as before: we can draw the part of the plum segment, which presents the difference of the two normalization moves before the reduction, out of the arcades because the move of both crossing points of the mal and plum segments does not change the colors of the crossed arcs and the loops of both points remain neighbours when crossing the arcs, as indicated by the green color in the diagram for the case $u_3 > u_1 > u_2$. So we get isomorphic *AFUs* generated by the normalization of $\mathcal{K}$ and $\mathcal{K}'$ .

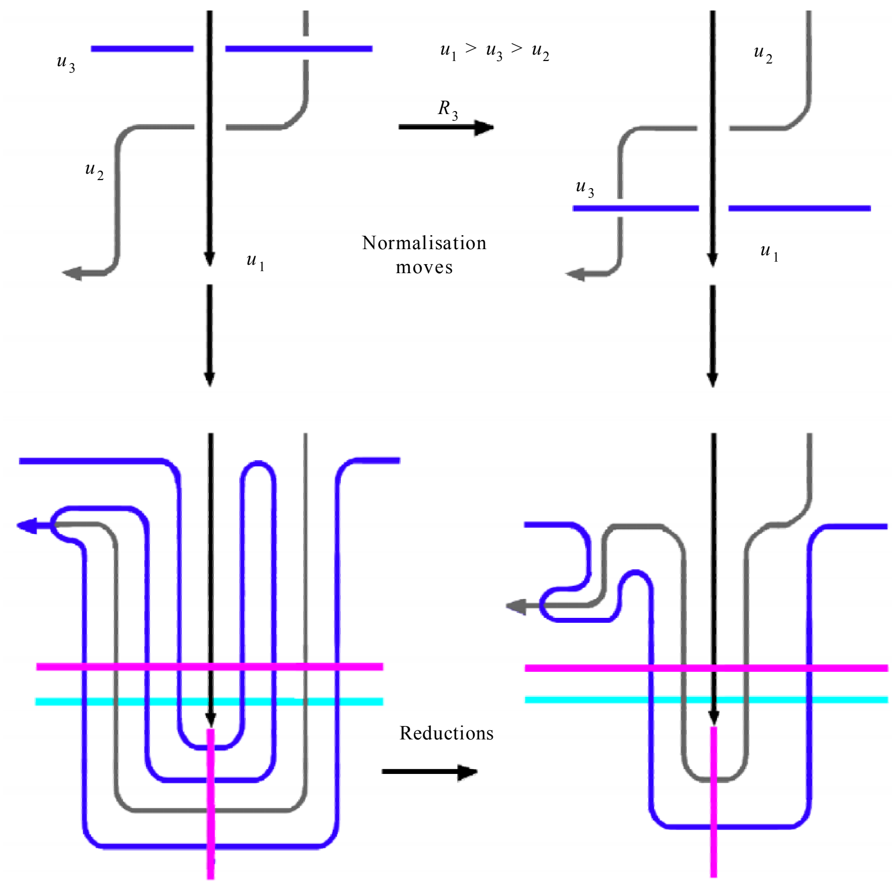



Fig. 12.

This means that in this case we get invariance of the results of the *a*-normalizations under the $R_3$-move, too.

**5.3. Case $u_3 > u_1 > u_2$**

The interpretation of the diagrams of the following figure is as before. But here we don't get invariance. A complete reduction of the moved $u_3$-loop is blocked only in one place: The neighbour parts of the loops, which we could reduce in the former cases completely, pass in the left lower diagram the vertical blue and the vertical red arc. The reduction is blocked by the point, where these arcs are connected and it is the only point where this happens. The reason is that the plum segment moved along the black segment is overcrossing and the black segment is ending.

We solve this problem in the following way: moving a loop in the normalization process, we don't first finish this move and then reduce, but in each case crossing during the move of an arc we test if a reduction is possible. If it is the case, we reduce. If it is not, we test if a "jump" over the arc would enable a reduction.

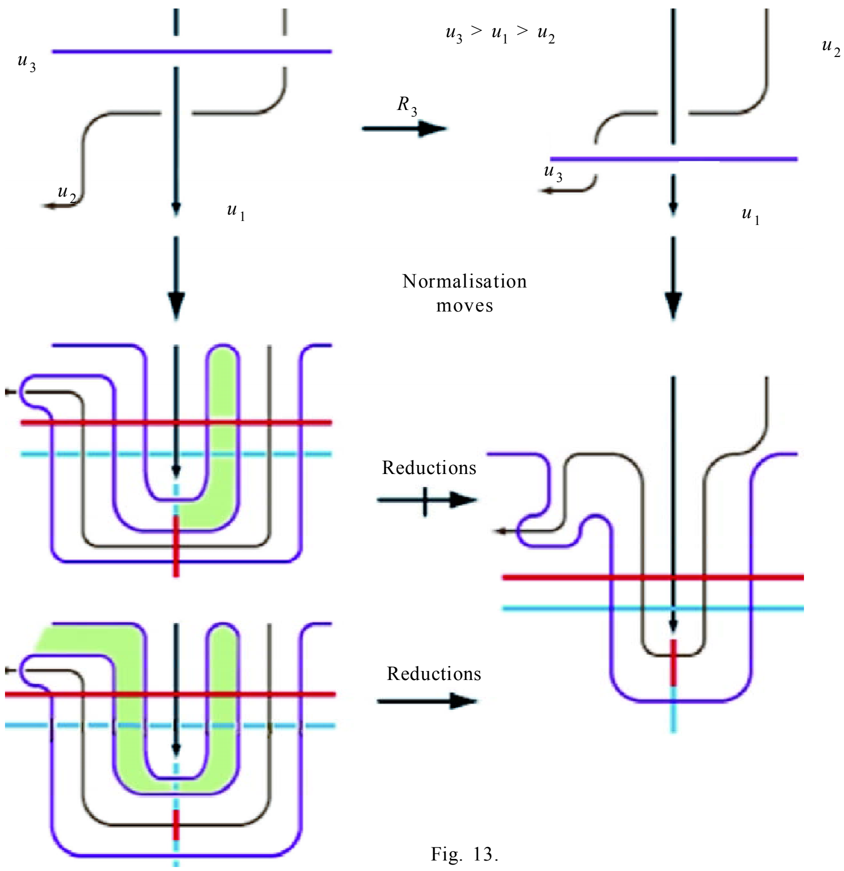



Fig. 13.

Formally this jump would consist in the construction of two new arcs; in our case in decomposing the red arc by two red arcs and one blue arc as represented by the diagram on the bottom of the left side of the figure. This makes a reduction possible. After having done this reduction we may proceed as usual with normalization moves. Having finished this move we reduce the arcs without crossing points. The result is then again invariance of the achieved *AFLs.*

We call this modification of the defined normalization steps an *extended normalization* step.

**Definition 2.** *An extended normalization is a sequence of extended normalization steps as just defined.*

With this modification of our normalizations we achieved the following result.

**Lemma 4**. *Let* $\mathcal{K}$ and $\mathcal{K}'$ *be two knot projections and*

$$R_3 : \mathcal{K} \to \mathcal{K}'$$

*a Reidemeister move. If* $(\alpha_{\mathcal{K}}, L_{\mathcal{K}})$ *and* $(\alpha_{\mathcal{K}'}, L_{\mathcal{K}'})$ *are achieved by extended a-normalizations, then both AFLs are isomorphic.*

It follows

**Theorem 3.** *To each knot projection* $\mathcal{K}$*, there exists a segment* **a** *on* $\mathcal{K}$ *such that the AFL achieved by an extended a-normalization is isomorphic to the extended* $a'$*-normalization of a minimal knot projection* $\mathcal{K}'$ *equivalent to* $\mathcal{K}$*.*

## 6. The Complexity of the Knot Problem

The size of the sequence of partial *AFL*s we construct by the normalization of a knot projection $\mathcal{K}$ could double with each normalization step. Figure 14 represents a diagram, which describes the normalization of two crossing points generated by two neighbour $R_1$-moves and it gives some plausibility to our statement. A normalization move of a crossing point $P$ only follows the loop generated by the normalization move of a point $P'$ just before if $P'$ has been generated by a $R_1$-move. But we finish each normalization step by a reduction. So in the example represented by Fig. 14 we will not get the double length by the extended normalization move of the upper crossing point even if we didn't apply reduction after the move of the lower point. Therefore the growth with complexity $O(2^n)$ with $n$ the size of $\mathcal{K}$ will

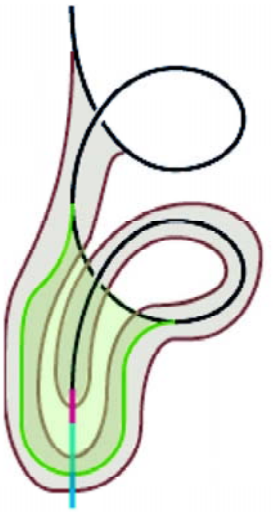


Fig. 14.

not happen. But we get as a trivial upper bound of the complexity of the extended normalization $O(2^n)$.

We get a sharper bound by the observation that can make the normalization moves irreducible by crossing both loops by a segment, which is atlternating over- and undercrossing. So we get $O\left(2^{\frac{1}{3}}\right)$ as bound for the normalization of a knot projection with $n$ crossing points. This bound may be sharp for the normalization of some minimal knot projections of size $n$ but it is not for projections of the same knot with size $m$ and $m - n$ large. To decide the equivalence of two different knot projections $\mathcal{K}, \mathcal{K}'$ we have to compute the extended a-normalizations for all the segments $a$ on $\mathcal{K}$ and $\mathcal{K}'$ and to test if there exists an isomorphic pair in these two sets. From the last theorem it follows that the two knot projections are only equivalent, if such a pair exists. In the case that $\mathcal{K}'$ is a circle it is one extended normalization sufficient for this decision. It follows

**Theorem 4.** *Let $\mathcal{K}$ and $\mathcal{K}'$ be two knot projections of size n respective to size m and $n \geq m$. Then the equivalence of $\mathcal{K}$ and $\mathcal{K}'$ can be decided in $O\left(n^2 \cdot 2^{\frac{n}{3}}\right)$ steps. If $\mathcal{K}'$ is a circle, then the problem can be decided in time $O\left(2^{\frac{n}{3}}\right)$.*

The factor $n$ comes in as the number $O(n)$ of different segments $a$ of $\mathcal{K}$. One can make this decision faster because it is sufficient to normalise the two knot projections only for one segment and to compute the other possible extended normalizations by *cyclic shifts* for both possible orientations of the knots. The alternating use of both orientations can substitute the extended normalization as follows from the discussion of $R_3$-moves in case $u_3 > u_1 > u_2$. It is open if this strategy leads to a more efficient algorithm to decide the equivalence problem.

Let be $\sigma_\alpha(\mathcal{K})$ the signature $\sigma_\alpha(L)$ of the *AFL* $(\alpha, L)$ generated by the extended normalization of $\mathcal{K}$ relative to the segment $a$ on $\mathcal{K}$. Then we may consider

$$\sum_a \beta_a \cdot \sigma_a(\mathcal{K})$$

with $\beta_a \in$ B, B := {0,1} a Boolean algebra as an invariant characterising the class of knot projections equivalent to $K$, uniquely.

## 7. Concluding Remarks

1. The complexity bound on the basis of the number of crossing points of knot projections leads to rough bounds. One gets better bounds, if one uses as base of the measure the sequence of the maximal crossing free segments of $\mathcal{K}$, which follow on a. The length of the loops generated by the normalization moves of points on the same segment have equal length. So, the complexity is exponentially not growing faster as with the number of these segments, because the complexity of the normalization moves of points on the same segment is for all these points equal. So, one should choose $a$ such that the number of these segments is minimal and one should choose the maximal segment as base for $a$. May be one would get a much faster algorithm, if we apply the normalization moves parallel to several segments $a$ to increase first the size of the simple segments.

2. The product $\sigma_\alpha(L) * \sigma_{\alpha'}(L')$ of the signatures of two *AFL*s is modulo $R_1$, $R_2$-reductions equal to a signature $\sigma_{\tilde{\alpha}}(\tilde{L})$ of an *AFL* $(\tilde{\alpha}, \tilde{L})$ representing the Schubert product of the knots represented by $(\alpha, L)$ and $(\alpha', L')$. The product of the signatures has to include a shift in the numbering of the arcades of the second factor.

4. If we are interested in weaker invariants as $(\alpha_{\mathcal{K}'}, L_{\mathcal{K}'})$, then during the normalization stronger reduction rules may be available, which will decrease the complexity. If $G$ is the direct product of the languages generated by the projections of our invariants on the free groups $S^*$ resp. $T^*$, then this is the case.

5. Lower bounds for the complexity of the knot problem are not known to me. But one can prove that the theory of topological nets [11] developed in connection with the algebraic representation of switching networks, which generalises the knot theory and, as it is used in physics [8],[9], is at least as hard as the Pressburger arithmetic [10].

*მათემატიკა & ინფორმატიკა*

# კვანძების პრობლემის ამოხსნის ეფექტური ალგორითმი

**გიუნტერ ჰოთცი**

*ზაარლენდის უნივერსიტეტი, გერმანია*

**ნაშრომში მოყვანილია ორი კვანძის ეკვივალენტურობის დადგენის ალგორითმი, რაც ამ ამოცანის ამოხსნადობას ამტკიცებს. ძირითადი იდეები ეყრდნობა კურტ რაიდემაისტერის მიერ 1930-იან წლებში შემოტანილი კვანძების ე.წ. AFL წარმოდგენას, რაც თავის მხრივ არის გაუსის წარმოდგენის მოდიფიკაცია. კვანძების AFL წარმოდგენით გვეძლევა გარკვეული ფორმალური ენის კონკრეტული სიტყვა, რომელიც მოცემულ კვანძს ცალსახად განსაზღვრავს. ამ ენის სიტყვებზე (და შესაბამისად კვანძების AFL წარმოდგენაზე) გარკვეული მანიპულაციები კვანძებზე რაიდემაისტერის მოძრაობების ჩატარების ტოლფასია. თუ მოცემულია ორი კვანძი K და K′, მოყვანილი ალგორითმის ბიჯების რაოდენობაა $O\left(n^2 \cdot 2^{\frac{n}{3}}\right)$, სადაც $n$ მეტია ან ტოლია K-სი და, შესაბამისად, მეტია ან ტოლი $O(n)$ კვანძების AFL წარმოდგენის წირების თვითთანაკვეთების რაოდენობისა. თუ K′ წრეწირის ეკვივალენტურია, მაშინ ეკვივალენტურობის დადგენა შესაძლებელია $O\left(2^{\frac{n}{3}}\right)$ ბიჯში.**

**სტატია სრულდება კვანძების ეკვივალენტურობის დადგენის პრობლემატიკის განხილვით.**

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