Mechanics

On Excitation of Orthotropic Half-Plane Caused by Concentrated Force, Moving on the Boundary

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ABSTRACT. In the paper, a problem of excitation of orthotropic half-plane caused by concentrated force, moving on the boundary is considered. It is shown that stress and wave range transmission in the anisotropic body have greater values than in isotropic materials. *© 2008 Bull. Georg. Natl. Acad. Sci.*

Key words: *orthotropic half-plane, anisotropy, isotropy, loading, stress.*

Let us consider a problem when a piecewise homogoneous body occupies a lower half-plane (*y*<0). Assume that at the moment $t = 0$ to the point of the half-plane $x = 0$ force $\vec{P}$ is instantly applied, which continues linear motion with a velocity $\vec{V}_1$. To determine stress state of the half-plane it is necessary to find components of tension $\sigma_x(x,y,t)$, $\sigma_y(x,y,t)$ and $\sigma_{xy}(x,y,t)$ which satisfy differential equations:

$$\left.\begin{aligned}\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}=\rho\frac{\partial^{2}y}{\partial t^2}\\ \frac{\partial t_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}=\rho\frac{\partial^{2}y}{\partial t^2}\end{aligned}\right\}, \tag{1}$$

where *u* and *v* are components of the displacement vector, satisfying zero initial and boundary conditions:

$$\left.\begin{aligned}\tau_{xy}&=0; & -\infty<x<\infty\\ \sigma_y&=\rho\sigma(x-v_1t); & -\infty<x<\infty\end{aligned}\right\}. \tag{2}$$

If the directions of the coordinate axes coincide with the main directions of an orthotropic body, then Hooke's law has the form [1, 2]:

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=l_{xx}=\frac{1}{E_1}(\sigma_x-\nu_1\sigma_y)\\ \frac{\partial v}{\partial y}&=l_{yy}=\frac{1}{E_2}(\sigma_y-\nu_2\sigma_x)\\ \frac{1}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)&=l_{xy}=\frac{1}{\mu}\tau_{xy}\end{aligned}\right\}, \tag{3}$$

where E_1 and E are absolute values of elasticity, ν_1 and ν – Poisson's ratios, μ – shear modulus.

In this case compatibility condition has a form:

$$\frac{1}{E_2}\left(\frac{\partial^2}{\partial x^2}-\nu_2\frac{\partial^2}{\partial y^2}\right)\sigma_y+\frac{1}{E_1}\left(\frac{\partial^2}{\partial y^2}-\nu_1\frac{\partial^2}{\partial x^2}\right)\sigma_x=\frac{2}{\mu}\frac{\partial^2\tau_{xy}}{\partial x\partial y}. \tag{4}$$

If we derive the first equation of the system (1) with respect to *x* and the second one with respect to *y*, subtract them from each other, and take into account system (3), we shall have [3]:

$$\left[\frac{\partial^2}{\partial x^2}-\rho\left(\frac{1}{E_1}+\frac{\nu_2}{E_2}\right)\frac{\partial}{\partial t^2}\right]\sigma_x=\frac{\partial^2}{\partial y^2}-\rho\left[\frac{1}{E_2}-\rho\left(\frac{1}{E_2}+\frac{\nu_1}{E_1}\right)\frac{\partial^2}{\partial t^2}\right]\sigma_y=0. \tag{5}$$

General solution of the equation (5) will have the form:

$$\left.\begin{aligned}\sigma_x&=\left[\frac{\partial^2}{\partial y^2}-\rho\left(\frac{1}{E_2}+\frac{\nu_1}{E_1}\right)\frac{\partial^2}{\partial t^2}\right]\Phi(x,y)\\ \sigma_y&=\left[\frac{\partial^2}{\partial x^2}-\rho\left(\frac{1}{E_1}+\frac{\nu_1}{E_2}\right)\frac{\partial^2}{\partial t^2}\right]\Phi(x,y)\end{aligned}\right\}. \tag{6}$$

If we add the same equations and take into account system (3), we shall have

$$\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_y}{\partial y^2}-\rho\frac{\partial^2}{\partial t^2}\left[\left(\frac{1}{E_1}-\frac{\nu_2}{E_2}\right)\sigma_x+\left(\frac{1}{E_2}-\frac{\nu_1}{E_1}\right)\sigma_y\right]=-2\frac{\partial^2 t_{xy}}{\partial x\partial y}.$$

Taking into account compatibility condition (4) in the last equation , we get

$$\left[\frac{1}{E_2}\frac{\partial^2\Phi}{\partial x^2}+\left(1-\frac{\nu_2}{E_2}\right)\frac{\partial^2}{\partial y^2}-\rho\frac{\partial^2}{\partial t^2}\left(\frac{1}{E_2}-\frac{\nu_1}{E_1}\right)\right]\sigma_y+\left[\left(1-\frac{\nu_1}{E_1}\right)\frac{\partial^2}{\partial x^2}+\frac{1}{E_1}\frac{\partial^2}{\partial y^2}-\rho\frac{\partial^2}{\partial t^2}\left(\frac{1}{E_1}-\frac{\nu_2}{E_2}\right)\right]\sigma_x=0. \tag{7}$$

Considering representations (6) in (4), we shall have:

$$\begin{aligned}&\frac{1}{E_2}\frac{\partial^4\Phi}{\partial x^4}+\frac{1}{E_1}\frac{\partial^4\Phi}{\partial y^4}+\frac{2\rho^2(1-\nu_1\nu_2)}{E_1E_2}\frac{\partial^4\Phi}{\partial t^4}+\left(2-\frac{\nu_1}{E_1}-\frac{\nu_2}{E_2}\right)\frac{\partial^4\Phi}{\partial x^2\partial y^2}-\\ &-\rho\left[\frac{2}{E_1}+\frac{1-\nu_2}{E_1E_2}+\frac{\nu_1}{E_1^2}-\frac{\nu_2^2}{E_2^2}\right]\frac{\partial^4\Phi}{\partial y^2\partial t^2}-\rho\left[\frac{2}{E_2}+\frac{1-\nu_1}{E_1E_2}+\frac{\nu_2}{E_2^2}-\frac{\nu_1^2}{E_1^2}\right]\frac{\partial^4\Phi}{\partial x^2\partial t^2}=0\end{aligned} \tag{8}$$

We reform variables $\xi=x-vt$, $\eta=y$ taking into account the fact that excitation extends with constant velocity. Putting them in (8), we get

$$A\frac{\partial^4\Psi}{\partial\xi^4}+B\frac{\partial^4\Psi}{\partial\psi^2\partial\eta^2}+C\frac{\partial^4\Psi}{\partial\eta^4}=0, \tag{9}$$

where $\Psi(\xi,\eta)=\Phi(x,y,t)$;

$$A+\frac{1}{E_2}+\frac{2\rho^2\nu_1^4(1-\nu_1\nu_2)}{E_1E_2}-\rho\nu_1^2\left(\frac{2}{E_2}-\frac{1-\nu_1}{E_1E_2}+\frac{\nu_2}{E_2^2}-\frac{\nu^2}{E_1^2}\right);$$

$$B=2-\frac{\nu_1}{E_1}-\frac{\nu_2}{E_2}-\rho\nu_1^2\left(\frac{2}{E_1}+\frac{1-\nu^2}{E_1E_2}+\frac{\nu_1}{E_1^2}-\frac{\nu_2^2}{E_2^2}\right);$$

$C = \frac{1}{E_1}.$

For components of tension we have

$$\left.\begin{aligned} \sigma_x &= \frac{\partial^2 \Psi}{\partial \eta^2} - \rho v_1^2 \left(\frac{1}{E_2} + \frac{\nu_1}{E_1} \right) \frac{\partial^2 \Psi}{\partial \xi^2} \\ \sigma_y &= \left[1 - \rho v_1^2 \left(\frac{1}{E_1} - \frac{\nu_2}{E_2} \right) \right] \frac{\partial^2 \Psi}{\partial \xi^2} \end{aligned} \right\}. \tag{10}$$

Taking into consideration the fact that for most of materials in nature, expression (9) is a differential equation of elliptic type [4]; its solution on half-plane $y<0$ may be presented by means of two analytic functions [5]:

$$\Psi(\xi, \eta) = 2\,\mathrm{Re}\left[F_1(z_1) + F_2(z_2) \right], \tag{11}$$

where $z_1 = \xi + i\beta_1\eta,\ z_2 = \xi + i\beta_2\eta,\ i\beta_1$ and $i\beta_2$ are roots of defining equation (9), respectively.

If we take into account equation (11) in the equation (10), we get:

$$\left.\begin{aligned} \sigma_x &= -2\,\mathrm{Re}\left[\left(\beta_1^2 + \rho v_1^2 \left(\frac{1}{E_2} + \frac{\nu_1}{E_1} \right) \right) F_1''(z_1) + \left(\beta_2^2 + \rho v_1^2 \left(\frac{1}{E_2} + \frac{\nu_1}{E_1} \right) \right) F_2''(z_2) \right] \\ \sigma_y &= 2\left[1 - \rho v_1^2 \left(\frac{1}{E_1} + \frac{\nu_2}{E_2} \right) \right] \mathrm{Re}\left[F_1''(z_1) + F_2''(z_2) \right] \end{aligned} \right\}. \tag{12}$$

Inserting (12) in equation (8), we shall have:

$$\left.\begin{aligned} u &= -\frac{2}{E_1} \left[(\beta_1^2 + a) F_1'(z_1) + (\beta_2^2 + a) F_2'(z_2) \right] \\ \sigma_y &= \frac{2}{E_2} \mathrm{Im}\left[(\nu_2 \beta_1^2 + b) F_1'(z_1) + (\nu_2 \beta_2^2 + b) F_2'(z_2) \right] \end{aligned} \right\}, \tag{13}$$

where $a = \frac{\rho v_1^2}{E_1}(1 - \nu_1 \nu_2) - \nu_1,\ b = \frac{\rho v_1^2}{E_1}(\nu_1 \nu_2 - 1) + 1.$

The third equality of system (3) gives

$$\tau_{xy} = \mathrm{Im}\left\{ \left[\frac{\beta_1}{E_1}(\beta_1^2 + a) + \frac{\nu_2 \beta_1^2 + b}{E_2} \right] F_1''(z_1) + \left[\frac{\beta_2}{E_1}(\beta_2^2 + a) + \frac{\nu_2 \beta_2^2 + b}{E_2} \right] F_2''(z_2) \right\}. \tag{14}$$

Considering boundary conditions (2), it is possible to define analytic functions F and F on the plane $y<0$ in such a way that they should satisfy the following boundary conditions:

$$\left.\begin{aligned} &2\left[1 - \rho v_1^2 \left(\frac{1}{E_1} + \frac{\nu_1}{E_2} \right) \right] \mathrm{Re}[F_1''(\xi) + F_2''(\xi)] = p\delta(\xi) \\ &\mathrm{Im}[rF_1''(\xi) + qF_2''(\xi)] = 0 \\ &\qquad -\infty < \xi < \infty \end{aligned} \right\}. \tag{15}$$

$$r=\frac{\beta_1}{E_1}(\beta_1^2+a)+\frac{v_2\beta_1^2+b}{E_2},\quad q=\frac{\beta_2}{E_2}(\beta_2^2+a)+\frac{v_2\beta_2^2+b}{E_2}.$$

The second equation of system (15) will be satisfied if we suppose that

$$F_2''(z)=-\frac{r}{q}F_1''(z). \tag{16}$$

Taking into consideration the first equality of (15) in the last equation, we get:

$$\operatorname{Re}F_1''(\xi)=\frac{p\delta(\xi)}{2\left[1-\rho v_1^2\left(\frac{1}{E_1}+\frac{v_2}{E_2}\right)\right]\left(1-\frac{r}{q}\right)}. \tag{17}$$

Supposing that the right part of equality (17) is known, we obtain Dirichlet's problem, the solution of which is given as follows:

$$F_1''(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{p}{2\left[1-\rho v_1^2\left(\frac{1}{E_1}+\frac{v_2}{E_2}\right)\right]\left(1-\frac{r}{q}\right)}\frac{d\xi}{z_1-z_2}d\xi.$$

Thus, we have:

$$\left.\begin{aligned}F_1''(z)&=\frac{1}{2\pi i\left[1-\rho v_1^2\left(\frac{1}{E_1}+\frac{v_2}{E_2}\right)\right]\left(1-\frac{r}{q}\right)}\cdot\frac{p}{z}\\ F_{21}''(z)&=\frac{1}{2\pi i\left[1-\rho v_1^2\left(\frac{1}{E_1}+\frac{v_2}{E_2}\right)\right]\left(1-\frac{q}{r}\right)}\cdot\frac{p}{z}\end{aligned}\right\} \tag{18}$$

Finally, for tension components we shall have the following expressions:

$$\left.\begin{aligned}\sigma_x&=\frac{py}{\pi(1-d)}\left[\frac{\beta_1(\beta_1^2+c)}{\left(1-\frac{r}{q}\right)\left[(x-v_1t)^2+\beta_1^2y^2\right]}+\frac{\beta_2(\beta_2^2+c)}{\left(1-\frac{q}{r}\right)\left[(x-v_1t)^2+\beta_2^2y^2\right]}\right],\\ \sigma_y&=\frac{p}{\pi}\left[\frac{\beta_1}{\left(\frac{r}{q}-1\right)\left[(x-v_1t)^2+\beta_1^2y^2\right]}+\frac{\beta_2}{\left(\frac{q}{r}-1\right)\left[(x-v_1t)^2+\beta_2^2y^2\right]}\right]\\ \tau_{xy}&=\frac{p(x-v_1t)}{2\pi(1-d)}\left[\frac{r}{\left(\frac{r}{q}-1\right)\left[(x-v_1t)^2+\beta_1^2y^2\right]}+\frac{q}{\left(\frac{q}{r}-1\right)\left[(x-v_1t)^2+\beta_2^2y^2\right]}\right]\end{aligned}\right\}, \tag{19}$$

where $d \equiv \rho v_1^2 \left(\frac{1}{E_1} + \frac{v_2}{E_2} \right)$, $\quad c \equiv \rho v_1^2 \left(\frac{1}{E_2} + \frac{v_1}{E_1} \right)$.

According to (19) we can conclude that anisotropy greatly influences the distribution of tensions in the body. In the anisotropic body values of tensions are much bigger than in isotropic materials, also, the range of wave transmission in them is noticeably expanded.

If in (16) we assume that $r=q$ and put it in (15), then, considering the values of $\vec{P}$, a boundary value problem can have a great number of solutions or, on the contrary, it will have no solution. In particular, if a body moves on the boundary with zero loading, then the problem has a great number of solutions; if effective loading differs from zero, then the problem has not a general solution.

It should be noted that the considered problem is analogous to the case with an isotropic body.

მექანიკა

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