Mathematics

Maximal and Potential Operators in Variable Morrey Spaces Defined on Nondoubling Quasimetric Measure Spaces

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ABSTRACT. The boundedness of modified maximal operator and potentials in variable Morrey spaces defined on quasimetric measure spaces, where the doubling condition is not needed, is established. © 2008 Bull. Georg. Natl. Acad. Sci.

Key words: maximal operator, potential, non-homogeneous spaces, variable Morrey space, boundedness.

Let $X = (X, \rho, \mu)$ be a topological space with a complete measure $\mu$ such that the space of compactly supported continuous functions is dense in $L^1(X, \mu)$ and there exists a non-negative real-valued function (quasimetric) $d$ on $X \times X$ satisfying the conditions:

(i) $d(x, y) = 0$ for all $x, y \in X$;
(ii) $d(x, y) > 0$ for all $x \neq y, x, y \in X$;
(iii) there exists a constant $a_1 > 0$, such that $d(x, y) \leq a_1(d(x, z) + d(z, y))$ for all $x, y, z \in X$;
(iv) there exists a constant $a_0 > 0$, such that $d(x, y) \leq a_0(d(y, x))$ for all $x, y \in X$.

We assume that the balls $B(a, r) : = \{x \in X: \rho(a, x) < r\}$ are measurable, for all $a \in X$ and $r > 0$, and $0 \mu(B(a, r)) < \infty$; for every neighborhood $V$ of $x \in X$, there exists $r > 0$, such that $B(x, r) \subset V$. We also suppose that $\mu(X) = \infty$ and $\mu\{a\} = 0$ for all $a \in X$.

The triple $(X, \rho, \mu)$ will be called quasimetric measure space. If $\mu$ satisfies the doubling condition

$$\mu(B(x, 2r)) \leq c\mu(B(x, r)),$$

where the positive constant $c$ does not depend on $x \in X$ and $r > 0$, then $(X, d, \mu)$ is called a space of homogeneous type (SHT). A quasimetric measure space, where the doubling condition might be failed, is also called a non-homogeneous space.

We say that the measure $\mu$ satisfies the growth condition ($\mu \in GC$) if there is a positive constant $b$ such that for all $a \in X$ and $r > 0$,

$$\mu(B(a, r)) \leq br \quad (1)$$

The boundedness of maximal and potential operators in Lebesgue spaces on non-homogeneous spaces was established in [1] (for Euclidean spaces), [2-5] (see also [6, Ch. 6]).

Suppose that $p$ is a $\mu$-measurable function on $X$. Denote
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\[ p_-(E) = \inf_E p ; \quad p_+(E) = \sup_E p \]

for a \( \mu \)-measurable set \( E \subset X \);

\[ p := p_-(X); \quad p := p_+(X). \]

\( \cdots \cdots \cdots \cdot p \leq p_+ < \infty \). The Lebesgue space with variable exponent \( L^{p_+}(X) \) (or \( L^{p_+}(\Omega) \)) is the class of all \( \mu \)-measurable functions \( f \) on \( X \) for which

\[ S_p(f) := \int_X |f(x)|^{p(x)} \mu(x) < \infty. \]

The norm in \( L^{p_+}(X) \) is defined as follows:

\[ \| f \|_{L^{p_+}(X)} = \inf \{ \lambda > 0 : S_p(f/\lambda) \leq 1 \}. \]

It is known (see e.g. [7-10]) that \( L^{p_+} \) space is a Banach space. For other properties of \( L^{p_+} \) we refer to [7,25] etc.

The boundedness of the Hardy-Littlewood maximal and potential operators in \( L^{p_+}(\Omega) \) (\( \Omega \subseteq \mathbb{R}^n \)) spaces was established in [12-15]. The same problem on an SHT was studied in [10,17-20] etc.

**Definition 1.** Let \( N \geq 1 \) be a constant. We say that \( p \notin P(N) \) if there is a positive constant \( C \) such that

\[ CNr_xB_r + r_xB_r p_x \leq C, \]

for all \( x \in X \) and \( r > 0 \).

Now we are ready to define variable exponent Morrey spaces.

**Definition 2.** Let \( N \geq 1 \) be a constant. Suppose that \( 1 < q_\leq q(x) \leq p(x) \leq p_+ < \infty \). We say that \( f \in M^{p(x)}_{\Omega}(X) \) if

\[ ||f||_{M^{p(x)}_{\Omega}(X)} = \sup_{x \in X, r > 0} \mu(B(x, N r))^{\frac{1}{p(x)}} \int_{B(x, r)} \mu(y)^{-\frac{1}{q(x)}} ||f||_{L^{q(x)}(B(x, r))} < \infty. \]

It is obvious that \( ||f||_{M^{p(x)}_{\Omega}(X)} = ||f||_{L^{q(x)}(X)} \) if \( p(x) = q(x) \).

For some properties of the spaces \( M^{p(x)}_{\Omega}(X) \), where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) see [21]. For variable exponent Morrey spaces on an SHT we refer e.g. [20].

The boundedness problem for maximal and fractional integrals in classical Morrey spaces \( (p = \text{const}, q = \text{const}) \) defined on Euclidean spaces was studied in [22-24]. The same problem for constant exponents in the case of quasimetric metric spaces was investigated in [5,26].

In [21] the boundedness of the Hardy-Littlewood maximal and Riesz potential operator in \( M^{p(x)}_{\Omega}(X) \) on a bounded domain \( \Omega \subset \mathbb{R}^n \) defined with respect to the Lebesgue measure was obtained. In [20] the authors have shown that maximal and Calderón-Zygmund operators on an SHT with finite measure and diameter are bounded in \( L^{p(x)}_{\Omega}(X) \) provided that \( p \) satisfies log-Hölder continuity condition on \( X \).

Let \( N \geq 1 \) be a constant and let

\[ M_f(x) = \sup_{x \in X, r > 0} \int_{B(x, r)} \frac{f(y) \mu(y)^{-\frac{1}{p(x)}}}{\mu(B(x, N r))^{\frac{1}{p(x)}}} < \infty; \]

\[ I_{\alpha(x)}f(x) = \int_X \frac{f(y)}{d(x, y)^{1-\alpha(x)}} d\mu(y), \quad x \in X, \quad 0 < \alpha \leq \alpha_x < 1, \]

be the modified maximal and fractional integral operators respectively on a quasimetric measure space \((X, d, \mu)\).
Now we formulate the main results of the paper.

**Theorem 1.** Let \(1 < p_+ \leq p_+ < \infty\) and let \(N:=a_1(1+2a_0)\), where \(a_0\) and \(a_1\) are from the definition of the quasimetric \(d\). If there exists a positive constant \(C\) such that for all \(x \in X\) and \(r > 0\), the inequality
\[
\mu(B(x, Nr)) p_+(\theta(x,r))^{-p(x)} p_-(x) \leq C
\]
holds, then \(M\) is bounded in \(L^{p_+}(X)\).

**Remark:** Notice that condition (2) implies condition (3).

To formulate the next results we need the notation
\[
a := a_1(a_0 + 1) + 1.
\]

**Theorem 2.** Let \(1 < q \leq q(x) \leq p_+ \leq p_+ < \infty\). Suppose that \(N:=a_1(1+2a_0)\) and \(p, q \in P(N)\). Then \(M\) is bounded from \(L^{p(x)}(X)\) to \(L^{q(x)}(X)\).

**Theorem 3.** Let \(N:=a_1(1+2a_0)\), \(1 < q \leq q(x) \leq p_+ \leq p_+ < \infty\), \(1 < t \leq t(x) \leq s(x) \leq s_+ < \infty\). Suppose that
\[
0 < \alpha < \frac{1}{p_-}, \quad s(x) = \frac{p(x)}{1-a(x)p(x)}, \quad \frac{t(x)}{s(x)} = \frac{q(x)}{p(x)}
\]
and that \(p, q, \alpha \in P(N)\). Suppose also that the measure \(\mu\) satisfies condition (1). Then the operator \(I_{\alpha(x)}\) is bounded from \(M^{p(x)}(X)\) to \(M^{q(x)}(X)\).

**Acknowledgement.** The authors were partially supported by the INTAS Grant No. 06-1000017-8792 and the Georgian National Science Foundation Grant No. GNSF/STO7/3-169.
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Received June, 2008