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Mathematics

Optimization of a Delay Variable Structure System with Mixed Intermediate Condition

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ABSTRACT. In this paper an optimal control problem for a two-stage variable structure control system is investigated, whose law of movement at the first stage is described by an ordinary differential equation, but the second stage is described by a delay differential equation. These two stages of the system are connected by a mixed intermediate condition. The necessary conditions of optimality are obtained: for optimal control in the form of maximum principle and for the optimal structure changing moment in the form of equality, containing the effect of the mixed intermediate condition. The general results for linear time-optimal control problem are concretized.

Key words: *optimization, variable structure system, necessary conditions of optimality, mixed intermediate condition.*

1. Introduction

Change of the structure of a system means that the system at some beforehand unknown moment may go over from one law of movement to another. Moreover, after changing the structure the initial condition of the system depends on its previous state. This joins them into a single system with variable structure. Assume that the change of the system structure has to take place at an a priori unknown moment of time. Such problems are important for various practical applications. For example, in economics the need arises to change invested capital at some unknown moment[1,2]. In engineering a controlled apparatus is to start from another controlled apparatus [3,4]. The problem when traffic laws of movement of body change at a beforehand unknown moment is considered in [5]. Investigation of variable structure optimal control problems with delay is one of the important directions of the optimal control theory. The delay factor may arise in many practical problems in connection with time spent on signal transmission. The present paper deals with the optimal control problem for systems which change their structure only once. In a special case, its first situation is described by an ordinary differential equation, and the second situation by an ordinary differential equation with constant delays in phase coordinates. These two situations are joined by the condition which we called mixed intermediate condition. This means that at the switching moment part of the initial functions and values of trajectories generally do not coincide with each other (are discontinuous), and the other values of the initial functions and trajectories coincide (are continuous). In the present paper the necessary conditions of optimality are obtained by the method described in [6]: for the optimal final and structure changing moments in the form of equality; for optimal control in the form of point-wise maximum principle. Moreover, general results are concretized for a linear variable structure time-optimal control problem. An illustrative example is considered. Finally, we note that optimal control problems for variable structure systems with delay and continuous and discontinuous initial condition are investigated in [1,2,6].

2. Problem Statement. Necessary Conditions of Optimality

Let R_x^n be the n-dimensional vector space of points $x=(x^1,...,x^n)^T$, T means transpose; $O \subset R_x^n$, $U_0 \subset R_u^r$, $V_0 \subset R_v^V$ be open sets; $t_0, \alpha, \beta, \gamma, \sigma, \tau > 0$, $\eta > 0$ be given numbers: $t_0 < \alpha < \beta < \gamma < \sigma$, $\alpha - t_0 \geq \eta$, $\gamma - \beta \geq \tau_0 = \max\{\tau, \eta\}$; further $y=(p,z)^T \in R_y^m$, $p \in R_p^k$, $z \in R_z^e$, $m=k+e$; $P \subset R_p^k$, $Z \subset R_z^e$ be open sets; the function $f(t,x,u) \in R_x^n$ is continuous on the set $[t_0,\beta] \times O \times U_0$ and continuously differentiable with respect to $x \in O$; the function $g(t,p,z,v) \in R_y^m$ is continuous on the set $[\alpha,\sigma] \times P \times Z \times V_0$ and continuously differentiable with respect to $(p,z) \in P \times Z$; $K_0 : O \to P$, $K_1 : O \to Z$ are continuously differentiable functions; Ω and Δ are sets of piecewise-continuous control functions $u:[t_0,\beta] \to U$ and $v:[\alpha,\sigma] \to V$ with a finite number of discontinuity points of first kind, where $U \subset U_0$, $V \subset V_0$ are given sets; finally the functions $q^0(t,y) \in R$, $q(t,y) \in R_q^l$ are continuously differentiable with respect to $t \in [\gamma,\sigma]$ and $y \in (P,Z)^T = \{y=(p,z)^T \in R_y^m : p \in P, z \in Z\}$.

To each element $\omega = (\theta, t_1, u(\cdot), v(\cdot)) \in W = (\alpha,\beta) \times (\gamma,\sigma) \times \Omega \times \Delta$ we assign a variable structure system

$$\begin{cases} \dot{x}(t) = f(t,x(t),u(t)), \ t_0 \leq t \leq \theta, \\ \dot{y}(t) = (\dot{p}(t),\dot{z}(t))^T = g(t,p(t-\tau),z(t-\eta),v(t)), \ \ \theta \leq t \leq t_1, \end{cases} \tag{1}$$

with the initial condition

$$x(t_0) = x_0 \tag{2}$$

and mixed intermediate initial condition at switching moment θ

$$\begin{cases} p(t) = \varphi(t), \theta - \tau \leq t < \theta, \ p(\theta) = K_0(x(\theta)), \ (3a) \\ z(t) = K_1(x(t)), \quad \theta - \eta \leq t \leq \theta, \qquad\qquad (3b) \end{cases} \tag{3}$$

where $x_0 \in O$ is a fixed point, $\varphi(t) \in P$, $t \in [\alpha - \tau, \beta]$ is given a continuous initial function. The condition (3a) is the discontinuous part of the mixed intermediate condition, because in general $\varphi(\theta) \neq K_0(x(\theta))$ and condition (3b) is continuous part of the mixed intermediate condition.

Definition 1. Let $\omega = (\theta, t_1, u(\cdot), v(\cdot)) \in W$. The pair of functions

$$\{x(t) = x(t;\omega), \ t \in [t_0,\theta]; \ \ y(t) = y(t;\omega) \in (P,Z)^T, \ t \in [\theta - \eta, t_1]\}$$

is called a solution corresponding to ω, if these functions satisfy conditions (2),(3), and the functions $\{x(t), t \in [t_0,\theta]; y(t), \ t \in [\theta, t_1]\}$ are absolutely continuous, satisfying the equations of the system (1) almost everywhere on the corresponding intervals.

Definition 2. The element $\omega \in W$ is called admissible, if the corresponding solution satisfies the boundary conditions

$$q(t_1, y(t_1)) = 0 \tag{4}$$

We denote the set of admissible elements by W_0.

Definition 3. The element $\omega_0 = (\theta_0, t_{10}, u_0(\cdot), v_0(\cdot)) \in W_0$ is called optimal, if for any $\omega \in W_0$ the following inequality

$$q^0(t_{10}, y_0(t_{10})) \leq q^0(t_1, y(t_1)), \tag{5}$$

is fulfilled, where $y_0(t) = (p_0(t), z_0(t))^T = y(t;\omega_0)$. We called (1)-(5) variable structure optimal control problem. The aim is to find the element ω_0.

Theorem 1. *Let ω_0 be an optimal element and the function $v_0(t)$ be continuous at point $\theta_0 + \eta$. Then there exist a vector $\pi = (\pi_0,...,\pi_l) \neq 0$, $\pi_0 \leq 0$ and a solution $\{\chi(t) = (\chi_1(t),...,\chi_n(t)), \ t \in [t_0,\theta_0];$ $\psi(t) = (\psi_0(t),\psi_1(t)) = (\psi_{01}(t),...,\psi_{0k},\psi_{11}(t),..., \psi_{1e}(t)), t \in [\theta_0, t_{10} + \tau_0]\}$ of the conjugate variable structure system*

$$\begin{cases} \dot{\chi}(t) = -\chi(t)\dfrac{\partial f_0[t]}{\partial x} - \mu(t)\psi(t+\eta)\dfrac{\partial g_0[t+\eta]}{\partial z}\dfrac{\partial K_1(x_0(t))}{\partial x}, \quad t_0 \le t \le \theta_0, \\ \begin{cases} \dot{\psi}_0(t) = -\psi(t+\tau)\dfrac{\partial g_0[t+\tau]}{\partial p}, \\ \dot{\psi}_1(t) = -\psi(t+\eta)\dfrac{\partial g_0[t+\eta]}{\partial z}, \quad \theta_0 \le t \le t_{10}, \\ \psi(t) = 0, \ t \in (t_{10}, t_{10}+\tau_0] \end{cases} \end{cases}$$

such that the following conditions are fulfilled:

1) the maximum principle

$$\chi(t) f_0[t] = \max_{u \in U} \chi(t) f(t, x_0(t), u), \ t_0 \le t \le \theta_0, \quad \psi(t) g_0[t] = \max_{v \in V} \psi(t) g(t, p_0(t-\tau), z_0(t-\eta), v), \ \theta_0 \le t \le t_{10};$$

2) the condition for the optimal switching moment θ_0

$$\chi(\theta_0) f_0[\theta_0] - \psi(\theta_0) g_0[\theta_0] - \psi(\theta_0+\tau)[g(\theta_0+\tau, K_0(x_0(\theta_0)), z_0(\theta_0-\eta+\tau), v_0(\theta_0+\tau)) - g(\theta_0+\tau, \varphi(\theta_0), x_0(\theta_0-\eta+\tau), v_0(\theta_0+\tau))] = 0;$$

3) the condition for the functions $\chi(t)$ *and* $\psi(t)$

$$\chi(\theta_0) = \psi(\theta_0)\frac{\partial K(x(\theta_0))}{\partial x}, \quad \psi(t_{10}) = \pi \frac{\partial Q_0}{\partial y};$$

4) the condition for the final moment t_{10}

$$\pi \frac{\partial Q_0}{\partial t_1} = -\psi(t_{10}) g_0[t_{10}].$$

Here $\mu(t)$ is the characteristic function of the interval $[\theta_0 - \eta, \theta_0]$; $K(x) = (K_0(x), K_1(x))^T$,

$$f_0[t] = f(t, x_0(t), u_0(t)), \ g_0[t] = g(t, p_0(t-\tau), z_0(t-\eta), v_0(t)), \ Q = (q^0, q)^T, \ \frac{\partial Q_0}{\partial y} = \frac{\partial}{\partial y} Q(t_{10}, y_0(t_{10})).$$

Remark 1. In condition 2) the member

$\psi(\theta_0+\tau)[g(\theta_0+\tau, K_0(x_0(\theta_0)), z_0(\theta_0-\eta+\tau), v_0(\theta_0+\tau)) - g(\theta_0+\tau, \varphi(\theta_0), z_0(\theta_0-\eta+\tau), v_0(\theta_0+\tau))]$

is the discontinuity effect of intermediate condition (3a) and the member $\psi(\theta_0) g_0[\theta_0]$ is the continuity effect of intermediate condition (3b). If $rank(\frac{\partial Q_0}{\partial t_1}, \frac{\partial Q_0}{\partial y}) = 1 + l$, then $\|\psi\| = \sup\{|\psi(t)| : \theta_0 \le t \le t_{10}\} \ne 0$.

3. Linear Time-Optimal Control Problem

Let us consider a linear time-optimal control problem. Assume that

$$f(t, x, u) = Ax + Bu, \ g(t, p, z, v) = Cp + Dz + Ev, \ K_0(x) = K_0 x, \ K_1(x) = K_1 x,$$

let $q^0(t_1, y(t_1)) = t_1 - t_0$, $q(t_1, y(t_1)) = y(t_1) - y_1$, where A, B, C, D, E, K_0, K_1 are constant matrices with appropriate dimensions; $y_1 \in (P, Z)^T$ is a fixed point. Theorem 2, formulated below, is the result of theorem 1.

Theorem 2. *Let* ω_0 *be an optimal element. Then there exists a solution*

$$\{\chi(t), \ t \in [t_0, \theta_0]; \ \psi(t) = (\psi_0(t), \psi_1(t)), t \in [\theta_0, t_{10} + \tau_0]\}$$

of the linear conjugate variable structure system

$$\begin{cases} \dot{\chi}(t) = -\chi(t) A - \mu(t)\psi(t+\eta) D K_1, \ t_0 \le t \le \theta_0, \\ \begin{cases} \dot{\psi}_0(t) = -\psi(t+\tau) C, \\ \dot{\psi}_1(t) = -\psi(t+\eta) D, \qquad \theta_0 \le t \le t_{10}, \\ \psi(t) = 0, \ t_{10} < t \le t_{10} + \tau_0 \end{cases} \end{cases}$$

such that $\|\psi\| \ne 0$ *and the following conditions are fulfilled:*

1) the maximum principle

$$\chi(t) B u_0(t) = \max_{u \in U} \chi(t) B u, \ t_0 \le t \le \theta_0, \ \psi(t) E v_0(t) = \max_{v \in V} \psi(t) E v, \ \theta_0 \le t \le t_{10};$$

2) the condition for $\chi(t)$

$$\chi(\theta_0)=\psi(\theta_0)\begin{pmatrix}K_0\\K_1\end{pmatrix};$$

3) the condition for the moment θ_0

$$\chi(\theta_0)[Ax_0(\theta_0)+Bu_0(\theta_0)]-\psi(\theta_0)[C\varphi(\theta_0-\tau)+DK_1x_0(\theta_0-\eta)+Ev_0(\theta_0)]-\psi(\theta_0+\tau)C[K_0x_0(\theta_0)-\varphi(\theta_0)]=0;$$

4) the condition for the moment t_{10}

$$\psi(t_{10})[Cp_0(t_{10}-\tau)+Dz_0(t_{10}-\eta)+Ev_0(t_{10})]\geq 0\,.$$

Example. We consider the variable structure system

$$\begin{cases}\dot{x}^1=x^2\\ \dot{x}^2=x^3\\ \dot{x}^3=u(t),\ t_0\leq t\leq\theta,u(t)\in U_0=[-1,1]\end{cases}$$

$$\begin{cases}\dot{p}(t)=z(t-\eta)\\ \dot{z}(t)=p(t-\tau)+v(t),\theta\leq t\leq t_1,v(t)\in V_0=[0,1]\end{cases}$$

the initial condition

$$x^1(t_0)=x^2(t_0)=x^3(t_0)=x^0$$

the mixed intermediate conditions:

$$p(t)=\varphi(t),\theta-\tau\leq t<\theta,\quad p(\theta)=x^2(\theta),$$
$$z(t)=x^3(t),\quad \theta-\eta\leq t\leq\theta$$

the conditions at the moment t_1 :

$$p(t_1)=p^1,\quad z(t_1)=z^1$$

the function of purpose

$$t_1-t_0\to\min.$$

Here $\varphi(t)\in R,\ t\in[\alpha-\tau,\beta]$ is a given continuous initial function; $x^0,\ p^1,\ z^1\in R$ are given numbers.

It is not difficult to see that in this example
$n=3, m=2,\ k=1, e=1, r=1, \nu=1;$

$$A=\begin{pmatrix}0\,1\,0\\0\,0\,1\\0\,0\,0\end{pmatrix},\ B=\begin{pmatrix}0\\0\\1\end{pmatrix},\ C=\begin{pmatrix}0\\1\end{pmatrix},\ D=\begin{pmatrix}1\\0\end{pmatrix},\ E=\begin{pmatrix}0\\1\end{pmatrix},\ K_0=(0,1,\,0),\ K_1=(0,0,1).$$

Let $\omega_0=(\theta_0,t_{10},u_0(\cdot),v_0(\cdot))$ be an optimal element. According to theorem 2 there exists a solution

$$\begin{cases}\chi(t)=(\chi_1(t),\chi_2(t),\chi_3(t)),\ t_0\leq t\leq\theta_0,\\ \psi(t)=(\psi_{01}(t),\psi_{11}(t)),\ \theta_0\leq t\leq t_{10}+\tau_0,\end{cases}$$

of the conjugate equation

$$\begin{cases}\dot{\chi}_1(t)=0\\ \dot{\chi}_2(t)=-\chi_1(t)\\ \dot{\chi}_3(t)=-\chi_2(t)-\mu(t)\psi_{01}(t+\eta),\ t_0\leq t\leq\theta_0,\end{cases}$$

$$\begin{cases}\dot{\psi}_{01}(t)=-\psi_{11}(t+\tau),\\ \dot{\psi}_{11}(t)=-\psi_{01}(t+\eta), & \theta_0\leq t\leq t_{10},\\ \psi_{01}(t)=\psi_{11}(t)=0, & \theta_0<t\leq t_{10}+\tau_0,\end{cases}$$

such that

$$\|\psi\|\neq 0 \tag{6}$$

the following conditions are fulfilled:
the maximum principle

$$u_0(t)=\begin{cases}-1, & for\ \ \chi_3(t)<0,\\ \ \ 1, & for\ \ \chi_3(t)>0;\end{cases}\qquad v_0(t)=\begin{cases}0, & for\ \psi_{11}(t)<0,\\ 1, & for\ \psi_{11}(t)>0;\end{cases}$$

the condition for the function $\chi(t) = (\chi_1(t), \chi_2(t), \chi_3(t))$:

$$\chi_1(\theta_0) = 0,\ \chi_2(\theta_0) = \psi_{01}(\theta_0),\ \ \chi_3(\theta_0) = \psi_{11}(\theta_0)\,; \tag{7}$$

the condition for the moment θ_0

$$(\chi_1(\theta_0) + \chi_2(\theta_0))x^0 + \chi_3(\theta_0)u_0(\theta_0) - \psi_{01}(\theta_0)x^3(\theta_0 - \eta) - \psi_{11}(\theta_0)[\varphi(\theta_0 - \tau) +$$
$$+ \nu_0(\theta_0)] - \psi_{11}(\theta_0 + \tau)[x_0^2(\theta_0) - \varphi(\theta_0)] = 0;$$

the condition for the moment t_{10}

$$\psi_{01}(t_{10})z_0(t_{10} - \eta) + \psi_{11}(t_{10})[p_0(t_{10} - \tau) + \nu_0(t_{10})] \geq 0.$$

Remark 2. It is clear that $\chi_1(t) = 0,\ t \in [t_0, \theta_0]$, $\chi_2(t) = \chi_2^0 = const$ (see (4), (7)) and $\|\psi_{11}\| \neq 0$ (see (5),(6)). According to the above-mentioned and condition (7) for the moment θ_0 we get

$$\psi_{01}(\theta_0)[x^0 - x_0^3(\theta_0 - \eta)] + \psi_{11}(\theta_0)[u_0(\theta_0) - \varphi(\theta_0 - \tau) - \nu_0(\theta_0)] -$$
$$- \psi_{11}(\theta_0 + \tau)[x_0^2(\theta_0) - \varphi(\theta_0)] = 0.$$

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თ. თადუმაძე, ა. არსენაშვილი

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(წარმოდგენილია აკადემიკოს გ. ხარატიშვილის მიერ)

ნაშრომში შესწავლილია ოპტიმალური მართვის ამოცანა ორსაფეხურიანი ცვლადსტრუქტურიანი სისტემისთვის, რომლის მოძრაობის კანონი პირველ საფეხურზე აღიწერება ჩვეულებრივი დიფერენციალური განტოლებით, ხოლო მეორე საფეხურზე — დაგვიანებულარგუმენტიანი განტოლებით. სისტემის ეს ორი მდგომარეობა დაკავშირებულია შერეული შუალედური პირობით. მიღებულია ოპტიმალურობის აუცილებელი პირობები. მათ შორის, არსებითი სიახლეა სისტემის სტრუქტურის ცვლილების მომენტის ოპტიმალურობის აუცილებელი პირობა, რომელიც შეიცავს შერეული შუალედური პირობის ეფექტს. ზოგადი შედეგი დაკონკრეტებულია სწრაფქმედების აზრით ოპტიმალური ამოცანისათვის.

REFERENCES

1. *A. Arsenashvili, I. Ramishvili, T. Tadumadze* (2006), Mem. on Differential Equations Math. Phys., **39**: 1-34.
2. *T. Tadumadze, A. Arsenashvili* (2005), Appl. and Comp. Mathematics, **4**, 1: 48-53.
3. *H.H. Hall, E.D. Zambelii* (1958), JET Propulsion, **28**, 7: 463-465.
4. *M.S. Troicki* (1965), Automatics and Telemechanics, **4**, 29: 745-750.
5. *M.E. Salukvadze, V.E. Gabisonia, A.I. Arsenashvili* (1990), Preprints of 11^{th} IFAC WORLD CONGRESS, **10**: 211-213.
6. *G.L. Kharatishvili, T.A. Tadumadze* (2007), J. Math. Sci. (N.Y.), **104**, 1:1-175.

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