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Mathematics

On Time Nonlocal Problems for Some Nonlinear Evolution Equations

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ABSTRACT. In the present paper nonclassical problems for nonlinear first order evolution equations in abstract spaces are considered. Nonclassical problem with time nonlocal initial condition is studied for the first order evolution equation with semicontinuous and monotonic nonlinear operator. In suitable spaces the existence of solution of the nonclassical problem is proved. An iteration algorithm for approximation of solution of the time nonlocal problem by solutions of classical problems is constructed. Under suitable conditions on nonlocal operators and given vector-functions in the initial conditions it is proved that the nonclassical problem possesses a unique solution and the sequence of solutions of the constructed classical problems converges to the solution of the time nonlocal problem. The application of the obtained general results for investigation of time nonlocal problems for nonlinear parabolic equations is considered. *© 2008 Bull. Georg. Natl. Acad. Sci.*

Key words: *nonlinear first order evolution equations, time nonlocal problems, weak solutions.*

Time nonlocal problems are nonclassical problems which are used for mathematical modeling of various physical, chemical and other processes, when it is impossible to determine the initial state of the dynamical system, but the dependence of the initial values of the unknown function on its values at later times is given. Time nonlocal problems are generalizations of problems periodic with respect to the time variable and can be treated as control problems with initial conditions. Time nonlocal problems arise in mathematical modeling of processes of radionuclides propagation in Stokes fluid [1], diffusion and flow in porous media [2-4].

The nonclassical problems with discrete nonlocal initial conditions for parabolic equations were considered and studied in [5]. Later on, linear time nonlocal problems for some equations of mathematical physics were studied in [6-9]. Time nonlocal problem for nonlinear Navier-Stokes equations with linear discrete and integral nonlocal initial conditions were considered in [10]. Further, nonclassical initial boundary value problems for nonlinear Navier-Stokes equations with nonlinear discrete-integral time nonlocal conditions were studied in [11].

The present paper is devoted to the investigation of time nonlocal problems for nonlinear evolution equations. We consider a nonclassical problem for the first order evolution equation in abstract spaces with semicontinuous and monotonic nonlinear operator and nonlinear nonlocal initial condition. In suitable spaces of vector-valued distributions we investigate a time nonlocal problem and prove the existence of solution of the nonclassical problem. Moreover, we construct an iteration algorithm which permits one to approximate a time nonlocal problem by classical problems. Under corresponding conditions on nonlocal operator we prove the uniqueness of solution of the time nonlocal problem and show that the sequence of solutions of the constructed classical problems converges to the solution of the nonclassical problem. We consider some applications of the obtained general results for investigation of time nonlocal problems for nonlinear parabolic equations.

For bounded domain $\Omega \subset \mathbf{R}^n$, $n \ge 1$, with Lipschitz boundary $\Gamma = \partial\Omega$ we denote by $L^p(\Omega)$ the space of functions in Ω integrable to the p-th power in the Lebesgue sense and let $W^{s,p}(\Omega)$, $s \in \mathbf{R}$, $s \ge 0$, $1 \le p \le \infty$, be the Sobolev space of order s based on $L^p(\Omega)$. For any Banach space X, $C^0([0,T];X)$ denotes the space of continuous vector-functions on $[0,T]$ with values in X, $L^p(0,T;X)$ is the space of such vector-functions $g:(0,T) \to X$ that $\|g(t)\|_X \in L^p(0,T)$, $1 \le p \le \infty$. We denote by $g' = dg/dt$ the generalized derivative of $g \in L^p(0,T;X)$ in the sense of distributions $D'((0,T);X)$ on $(0,T)$ with values in X.

Let H be Hilbert space and V_i ($i = \overline{1,n}$, $n \in \mathbf{N}$) be reflexive Banach spaces, which are dense in H and are continuously embedded in it. We denote by $V = \bigcap_{i=1}^{n} V_i$ the space equipped with the norm $\|v\|_V = \sum_{i=1}^{n} \|v\|_{V_i}$, for all $v \in V$, and assume that V is separable space dense in H. By identifying the space H with its dual by using the scalar product in H we obtain $V \subset H \subset V'$ and $V_i \subset H \subset V_i'$ with continuous and dense embeddings, where V', V_i' are dual spaces of V and V_i, respectively ($i = \overline{1,n}$). Let A_i be nonlinear operators from V_i to its dual space V_i', which are semicontinuous and monotonic operators, i.e. the function $\left\langle A_i(v + yw), w^1 \right\rangle_i$ of the variable y is continuous and $\left\langle A_i(v) - A_i(w), v - w \right\rangle_i \ge 0$, for all $v, w, w^1 \in V_i$, where $\langle .,. \rangle_i$ denotes the duality relation between the spaces V_i' and V_i, $i = \overline{1,n}$. In addition, we suppose that the operators $A_i : V_i \to V_i'$ satisfy the following conditions $\|A_i(v)\|_{V_i'} \le c_{A_i} \|v\|_{V_i}^{p_i+1}$, $\langle A_i(v), v \rangle_i \ge \overline{c}_{A_i} [v]_{V_i}^{p_i+2}$, c_{A_i}, $\overline{c}_{A_i} = const > 0$, $p_i \in \mathbf{R}$, $p_i \ge 0$, for all $v \in V_i$, where $[.]_{V_i}$ denotes a seminorm in the space V_i for which there exists λ_i such that $[.]_{V_i} + \lambda_i \|.\|_H$ is equivalent to the norm $\|.\|_{V_i}$ in the space V_i, $i = \overline{1,n}$. Note that under the latter conditions the nonlinear operators A_i, $i = \overline{1,n}$, are continuous if the space V_i is considered with respect to the strong topology and the dual space V_i' with respect to the weak topology.

Let us consider a time nonlocal problem for a nonlinear first order evolution equation: find $u \in U = \bigcap_{i=1}^{n} L^{p_i+2}(0,T;V_i) \cap L^\infty(0,T;H)$, which satisfies the equation

$$\frac{d}{dt}(u(.),v)_H + \sum_{i=1}^{n} \left\langle A_i(u), v \right\rangle_i = \sum_{i=1}^{n} \left\langle f_i, v \right\rangle_i, \qquad \forall v \in V, \tag{1}$$

in the sense of distributions on $(0,T)$ together with the following nonlocal initial condition

$$u(0) = \sum_{j=1}^{m} g_j(u)(u(T_j),h_j)_H h_j^1 + \sum_{j=1}^{m} \int_0^T [\hat{g}_j(u)](\tau)(u(\tau),\hat{h}_j(\tau))_H \hat{h}_j^1(\tau) d\tau + u_0, \tag{2}$$

where $h_j, h_j^1 \in H$, $\hat{h}_j, \hat{h}_j^1 \in L^\infty(0,T;H)$, $\|h_j\|_H \le 1$, $\|h_j^1\|_H \le 1$, $\|\hat{h}_j(t)\|_H \le 1$, $\|\hat{h}_j^1(t)\|_H \le 1$, for almost all $t \in [0,T]$, $0 < T_j \le T$, $j = \overline{1,m}$, $g_j : U \to \mathbf{R}$ are weakly continuous and $\hat{g}_j : U \to L^1(0,T)$ are continuous when U is equipped with weak-* topology and the space $L^1(0,T)$ is considered with respect to the strong topology, i.e. if $\{w_N\}_{N \ge 1}$ converges to w weakly in each of the spaces $L^{p_i+2}(0,T;V_i)$, $i = \overline{1,n}$, and weakly-* in the space $L^\infty(0,T;H)$, then $g_j(w_N) \to g_j(w)$ and $\hat{g}_j(w_N) \to \hat{g}_j(w)$ strongly in the space $L^1(0,T)$, as $N \to \infty$, $j = \overline{1,m}$. Note that each operator A_i can be considered as operator from $L^{p_i+2}(0,T;V_i)$ to the space $L^{p_i^*}(0,T;V_i')$, $p_i^* = (p_i+2)/(p_i+1)$, $i = \overline{1,n}$. Therefore, if $f_i \in L^{p_i^*}(0,T;V_i')$, $i = \overline{1,n}$, then $u' \in \sum_{i=1}^{n} L^{p_i^*}(0,T;V_i') \subset L^{p^*}(0,T;V')$, $p^* = \min_{1 \le i \le n} p_i^*$, and since $u \in U \subset L^{p+2}(0,T;V)$, $p = \max_{1 \le i \le n}\{p_i\}$, applying the embedding theorem we obtain that $u \in C^0([0,T];H)$, which permits one to interpret the initial condition (2) as equality in the space H.

For time nonlocal problem (1), (2) the following existence theorem is proved.

Theorem 1. *If* $u_0 \in H$, $f_i \in L^{p_i^*}(0,T;V_i')$, $i=\overline{1,n}$, *and there exists positive constant* $\zeta < \sum_{i=1}^{n} \bar{c}_{A_i} \left(\inf_{v \in V_i} \frac{[v]_{V_i}}{\|v\|_H} \right)^2$, *when* $p_1 = p_2 = ... = p_n = 0$ *or arbitrary* $\zeta \in \mathbf{R}$, *when at least one* $p_i > 0$, *such that the following inequality is valid*

$$\sum_{j=1}^{m} (\sup\{| g_j(w) |: w \in C^0([0,T];V)\} e^{-\zeta T_j} + \int_0^T \sup\{| \hat{g}_j(w) | (\tau): w \in C^0([0,T];V)\} e^{-\zeta \tau} d\tau) < 1,$$

then the nonclassical problem for nonlinear first order evolution equation (1) with nonlocal initial condition (2) possesses the solution $u \in U$.

For practical solution of nonlinear time nonlocal problem (1), (2) one can construct an algorithm for approximation of the problem by a sequence of classical problems. Let us consider the following sequence of classical problems for the first order nonlinear evolution equation: find a vector-function $w_N \in U$, $N \geq 1$, which satisfies the equation in the sense of distributions on $(0,T)$ and the initial condition

$$\frac{d}{dt}(w_N(.),v)_H + \sum_{i=1}^{n} \langle A_i(w_N), v \rangle_i = \sum_{i=1}^{n} \langle f_i, v \rangle_i, \qquad \forall v \in V, \tag{3}$$

$$w_N(0) = \sum_{j=1}^{m} g_j(w_{N-1})(w_{N-1}(T_j), h_j)_H h_j^1 + \sum_{j=1}^{m} \int_0^T [\hat{g}_j(w_{N-1})](\tau)(w_{N-1}(\tau), \hat{h}_j(\tau))_H \hat{h}_j^1(\tau) d\tau + u_0, \tag{4}$$

where $w_0 \equiv 0$. For arbitrary operators $g_j : U \to \mathbf{R}$ and $\hat{g}_j : U \to L^1(0,T)$ ($j=\overline{1,m}$) in the nonlocal conditions classical problem (3), (4) possesses a unique solution [12] for each $n \in \mathbf{N}$. The following theorem gives conditions on g_j and $\hat{g}_j$ ($j=\overline{1,m}$), ensuring convergence of the sequence of solutions $\{w_N\}_{N\geq 1}$ of the classical problems to the solution of the time nonlocal problem.

Theorem 2. *If the operators* g_j *and* $\hat{g}_j$ ($j=\overline{1,m}$) *in the nonlocal conditions (2) satisfy the following conditions*

$$| g_j(w) - g_j(w^1) | \leq c_{g_j} \left\| w - w^1 \right\|_{C^0([0,T];H)}, \quad | [\hat{g}_j(w)](t) - [\hat{g}_j(w^1)](t) | \leq c_{\hat{g}_j} \left\| w(t) - w^1(t) \right\|_H,$$

for almost all $t \in (0,T)$ *and for all* $w, w^1 \in C^0([0,T];H)$, c_{g_j}, $c_{\hat{g}_j} = const > 0$, $j=\overline{1,m}$, *and*

$$\sum_{j=1}^{m} (\sup\{| g_j(w) |: w \in C^0([0,T];H)\} + \int_0^T \sup\{| \hat{g}_j(w) | (\tau): w \in C^0([0,T];H)\} d\tau) < 1,$$

then there exist neighborhood of zero D_0 *in the space* H *and neighborhoods of zero* D_i *in the spaces* $L^{p_i^*}(0,T;V_i')$ *such that, for* $u_0 \in D_0$ *and* $f_i \in D_i$ ($i=\overline{1,n}$) *nonclassical problem (1), (2) possesses a unique solution* $u \in U$ *and the sequence of solutions* $\{w_N\}_{N\geq 1}$ *to problems (3), (4) of the constructed iteration algorithm tends to the solution* u *in the space* $C^0([0,T];H)$.

Let us consider now some applications of the obtained general results for time nonlocal problems in the case of parabolic equations. We study a time nonlocal problem for the following nonlinear parabolic equation

$$\frac{\partial u}{\partial t} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(a_j \left| \frac{\partial u}{\partial x_j} \right|^p \frac{\partial u}{\partial x_j} \right) = f, \qquad (x,t) \in \Omega \times (0,T), \tag{5}$$

with mixed Dirichlet-Neumann boundary conditions

$$u(x,t) = 0, \quad (x,t) \in \Gamma_0 \times (0,T), \qquad \sum_{j=1}^{n} a_j \left| \frac{\partial u}{\partial x_j} \right|^p \frac{\partial u}{\partial x_j} \nu_j = \sigma, \qquad (x,t) \in \Gamma_1 \times (0,T), \tag{6}$$

and the following nonclassical initial condition

$$u(x,0)=\sum_{j=1}^{m}\sin^{k_j}\left(\int_0^T\int_\Omega u(\alpha,t)\eta_j(\alpha,t)d\alpha dt\right)\left(\int_\Omega u(\alpha,T_j)\xi_j(\alpha)d\alpha\right)\xi_j^1(x)+$$

$$+\sum_{j=1}^{m}\int_0^T\hat{\eta}_j(t)\left(\int_\Omega u(\alpha,t)\hat{\xi}_j(\alpha,t)d\alpha\right)\hat{\xi}_j^1(x,t)dt+u_0(x),\qquad x\in\Omega, \tag{7}$$

where $k_j\in\mathbf{N}\cup\{0\}$, $j=\overline{1,m}$, $\Gamma=\partial\Omega=\Gamma_0\cup\Gamma_{01}\cup\Gamma_1$, $\Gamma_0\cap\Gamma_1=\Gamma_{01}$, is a Lipschitz dissection of the boundary Γ, $\nu=(\nu_i)_{i=1}^n$ is the unit outward normal to $\Gamma=\partial\Omega$, f,σ,u_0 and ξ_j, ξ_j^1, η_j, $\hat{\xi}_j$, $\hat{\xi}_j^1$, $\hat{\eta}_j$ ($j=\overline{1,m}$) are given functions from suitable spaces. Note that, if all the functions in equation (5) are smooth enough, then multiplying both sides of equation (5) by arbitrary smooth function v vanishing on Γ_0, integrating over domain Ω and applying Green identities we obtain

$$\int_\Omega\frac{\partial u}{\partial t}(x,t)v(x)dx+\sum_{j=1}^{n}\int_\Omega a_j(x)\left|\frac{\partial u}{\partial x_j}\right|^p\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_j}dx=\int_\Omega f(x,t)v(x)dx+\int_{\Gamma_1}\sigma(x,t)v(x)dx\,. \tag{8}$$

Moreover, it can be proved that each smooth enough solution u of equation (8) is a solution of equation (5) and satisfies boundary conditions (6). So, in the spaces of smooth enough functions the problem (5)-(7) is equivalent to equation (8) with initial condition (7), and, consequently, the weak solution of the problem (5)-(7) can be defined as follows: find $u\in L^{p+2}(0,T;W_{\Gamma_0}^{1,p+2}(\Omega))\cap L^\infty(0,T;L^2(\Omega))$, $p\ge 0$, which satisfies the equation

$$\frac{d}{dt}(u(.),v)_{L^2(\Omega)}+a(u(.),v)=(f,v)_{L^2(\Omega)}+\left\langle\sigma,tr_{\Gamma_1}(v)\right\rangle_{\Gamma_1},\qquad\forall v\in W_{\Gamma_0}^{1,p+2}(\Omega),$$

in the sense of distributions in $(0,T)$ and the initial conditions (7), where $W_{\Gamma_0}^{1,p+2}(\Omega)=\{v\in W^{1,p+2}(\Omega);tr_{\Gamma_0}(v)=0\}$, $f\in L^{p'}(0,T;L^{p'}(\Omega))$, $\sigma\in L^{p'}(0,T;W^{-1/p',p'}(\Gamma_1))$, $p'=(p+2)/(p+1)$, $\langle.,.\rangle_{\Gamma_1}$ is the duality relation between $W^{-1/p',p'}(\Gamma_1)$ and $W^{1/p',p+2}(\Gamma_1)$, $W^{-1/p',p'}(\Gamma_1)$ is the dual space of $W^{1/p',p+2}(\Gamma_1)$, which is the space of traces on Γ_1 of the functions from $W^{1,p+2}(\Omega)$, $tr_{\Gamma_1}:W^{1,p+2}(\Omega)\to W^{1/p',p+2}(\Gamma_1)$ is the trace operator, $a(v^1,v)=\sum_{j=1}^{n}\int_\Omega a_j(x)\left|\frac{\partial v^1}{\partial x_j}\right|^p\frac{\partial v^1}{\partial x_j}\frac{\partial v}{\partial x_j}dx$, for all $v^1,v\in W^{1,p+2}(\Omega)$. If $a_j\ge c_a$, $c_a=const>0$, $a_j\in L^\infty(\Omega)$ ($j=\overline{1,m}$), then the operator A, corresponding to the form $a(.,.)$, $a(v^1,v)=\left\langle Av^1,v\right\rangle_\Omega$, is semicontinuous and monotonic from $W_{\Gamma_0}^{1,p+2}(\Omega)$ to its dual space $[W_{\Gamma_0}^{1,p+2}(\Omega)]'$, $\langle.,.\rangle_\Omega$ is the duality relation between $[W_{\Gamma_0}^{1,p+2}(\Omega)]'$ and $W_{\Gamma_0}^{1,p+2}(\Omega)$, and A satisfies the following conditions $\|Av\|_{[W_{\Gamma_0}^{1,p+2}(\Omega)]'}\le c_1\|v\|_{W^{1,p+2}(\Omega)}^{p+1}$,

$$\left\langle Av,v\right\rangle_\Omega=a(v,v)=\sum_{j=1}^{n}\int_\Omega a_j(x)\left|\frac{\partial v}{\partial x_j}\right|^p\frac{\partial v}{\partial x_j}\frac{\partial v}{\partial x_j}dx\ge c_a\sum_{j=1}^{n}\int_\Omega\left|\frac{\partial v}{\partial x_j}\right|^{p+2}dx=c_a[v]_{W^{1,p+2}(\Omega)}^{p+2}.$$

Therefore we can use Theorem 1 to obtain the corresponding result for the time nonlocal problem (5)-(7).

Theorem 3. *If* $u_0\in L^2(\Omega)$, $f\in L^{p'}(0,T;L^{p'}(\Omega))$, $\sigma\in L^{p'}(0,T;W^{-1/p',p'}(\Gamma_1))$, $\xi_j,\xi_j^1\in L^2(\Omega)$, $\eta_j\in L^1(0,T;L^2(\Omega))$, $\hat{\xi}_j,\hat{\xi}_j^1\in L^\infty(0,T;L^2(\Omega))$, $\hat{\eta}_j\in L^1(0,T)$, and the following condition is fulfilled

$$\sum_{j=1}^{m}\left[\left(\int_\Omega|\xi_j(x)|^2\,dx\right)^{1/2}\left(\int_\Omega|\xi_j^1(x)|^2\,dx\right)^{1/2}+\int_0^T\hat{\eta}_j(t)\left(\int_\Omega|\hat{\xi}_j(x,t)|^2\,dx\right)^{1/2}\left(\int_\Omega|\hat{\xi}_j^1(x,t)|^2\,dx\right)^{1/2}dt\right]<1,$$

then the time nonlocal problem for nonlinear parabolic equation (5) with nonclassical initial condition (7) is solvable in the space $L^{p+2}(0,T;W_{\Gamma_0}^{1,p+2}(\Omega))\cap L^\infty(0,T;L^2(\Omega))$.

Note that a similar result can be obtained for a parabolic equation with different powers in the nonlinear term

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i} \frac{\partial u}{\partial x_i} \right) = f\,, \qquad (x,t) \in \Omega \times (0,T)\,. \tag{9}$$

Time nonlocal problem for equation (9) with initial condition (7) and homogeneous boundary condition $u(x,t) = 0$ on $\Gamma \times (0,T)$ is a particular case of abstract problem (1), (2), where $H = L^2(\Omega)$, V_i is the closure of the set $D(\Omega)$ of infinitely differentiable functions with compact support in Ω in anisotropic Sobolev space $W_i^{1,p_i+2}(\Omega) = \{v \mid v \in L^{p_i+2}(\Omega), \partial v / \partial x_i \in L^{p_i+2}(\Omega)\}$, $p_i \geq 0$, $i = 1,2,...,n$, the seminorms $[.]_i$ are given by $[v]_i = \|\partial v / \partial x_i\|_{L^{p_i+2}(\Omega)}$, $i = \overline{1,n}$, and we can use the general results obtained for the abstract time nonlocal problems.

მათემატიკა

დროით არალოკალური ამოცანების შესახებ ზოგიერთი არაწრფივი ევოლუციური განტოლებისათვის

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ი. ჯავახიშვილის თბილისის სახელმწიფო უნივერსიტეტი

(წარმოდგენილია აკადემიის წევრის რ. ბანცურის მიერ)

წარმოდგენილ ნაშრომში განხილულია არაკლასიკური ამოცანები არაწრფივი პირველი რიგის განტოლებებისათვის. შესწავლილია არაკლასიკური ამოცანა დროით არალოკალური საწყისი პირობით პირველი რიგის ევოლუციური განტოლებისათვის სემიუწყვეტი და მონოტონური ოპერატორით. სათანადო სივრცეებში დამტკიცებულია არაკლასიკური ამოცანის ამონახსნის არსებობა. აგებულია დროით არალოკალური ამოცანის ამონახსნის კლასიკური ამოცანების ამონახსნებით აპროქსიმაციის იტერაციული ალგორითმი. საწყის პირობებში შემავალ არალოკალურ ოპერატორებზე და მოცემულ ფუნქციებზე შესაბამის პირობებში დამტკიცებულია, რომ არაკლასიკურ ამოცანას გააჩნია ერთადერთი ამონახსნი და აგებული კლასიკური ამოცანების ამონახსნების მიმდევრობა მიისწრაფვის დროით არალოკალური ამოცანის ამონახსნისაკენ. განხილულია მიღებული ზოგადი შედეგების გამოყენება არაწრფივი პარაბოლური განტოლებებისათვის დროით არალოკალური ამოცანების გამოსაკვლევად.

REFERENCES

1. *V.V. Shelukhi* (1993), Dynamics of Fluids With Free Boundaries: Inst. of Hydrodynamics, **107**: 180-193.
2. *C.V. Pao* (1995), J. Math. Anal. Appl., **195**, 3: 702-718.
3. *R.E. Ewing, R.D. Lazarov, Y. Lin* (2000), Computing, **64**: 157-182.
4. *R.E. Ewing, R.D. Lazarov, Y. Lin* (2000), Numer. Meth. Partial Diff. Eq., **16**: 285-311.
5. *D. Gordeziani* (1989), Rep. of Enlarged Sess. of the Sem. of I. Vekua Inst. Appl. Math., **4**: 57-60.
6. *G. Avalishvili* (2002), J. Appl. Anal., **8**, 2: 245-259.
7. *D. Gordeziani, H. Meladze, G. Avalishvil* (2003), J. Comput. Appl. Math., **88**: 66-78.
8. *D.G. Gordeziani, G.A. Avalishvili* (2005), Diff. Eq., **41**, 5: 703-711.
9. *D.G. Gordeziani, G.A. Avalishvili* (2005), Diff. Eq., **41**, 6: 852-859.
10. *D.G. Gordezian* (1993), Rep. of Enlarged Sess. of the Sem. of I. Vekua Inst. Appl. Math., **8**, 3: 17-19.
11. *D. Gordeziani, M. Avalishvili, G. Avalishvili* (2002), Appl. Math. Inform., **7**, 2: 66-77.
12. *J.-L. Lions* (1969), Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Paris.

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