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Mathematics

Lattice Isomorphisms of Nilpotent of Class 2 Hall's *W*-Power Groups

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(Presented by Academy Member H. Inassaridze)

ABSTRACT. **The lattice isomorphisms of 2-nilpotent Hall's W-groups are studied. It is shown that if a group contains the proper non-abelian W-subgroup, then the lattice isomorphism is induced by the semilinear isomorphism.** *© 2008 Bull. Georg. Natl. Acad. Sci.*

Key words: W-*power groups, nilpotent, lattice.*

The notion of a discrete W-power group was introduced by F. Hall [1]. For such groups he generalized some results from the theory of nilpotent groups. The importance of W-power nilpotent groups in the general theory of abstract groups is due to the fact that any torsion-free finitely generated nilpotent group is embedded into some W-power group.

On the other hand, the study of the properties of (universal) algebras, in particular groups, Lie algebras, rings etc. can be carried out taking into consideration the subalgebra lattices.

The aim of the present work is to study Hall's W-power groups from the lattice standpoint and to establish a relationship between the structure of a W-power group G and the structure of the lattice of all its subgroups.

In [2, 4] the fundamental theorems of affined and projective geometries for W-power groups were proved.

Let G and G_1 be W-power groups over the rings W and W_1 respectively. The bijection $f: G \to G_1$ will be called a semilinear isomorphism with respect to the isomorphism $\sigma: W \to W_1$ if the equality

$$f(x_1^{\alpha_1} x_2^{\alpha_2}) = f(x_1)^{\sigma(\alpha_1)} f(x_2)^{\sigma(\alpha_2)}$$

is fulfilled for any $x_1, x_2 \in G$ and $\alpha_1, \alpha_2 \in W$ and f will be called a semilinear antiisomorphism if the equality

$$f(x_1^{\alpha_1} x_2^{\alpha_2}) = f(x_2)^{\sigma(\alpha_2)} f(x_1)^{\sigma(\alpha_1)}$$

is fulfilled for any $x_1, x_2 \in G$ and $\alpha_1, \alpha_2 \in W$.

We will make use of generally accepted terminology (see [1, 4, 5]).

Notions. $L(G)$ is the lattice of all W-subgroups of the W-power group G over the ring W.

$\varphi: L(G) \to L(G_1)$ lattice isomorphism. $A^{\varphi} \leq G^{\varphi}$ will denote the image of the subgroup $A \subseteq G$.

$[A,A]$, $N(A)$ will denote the commutator and normalizer of the W-subgroup $A \subseteq G$ respectively;

$Z(A)$ is the center of $A \leq G$, $<X>$ denotes the subgroup generated by the set $X \subseteq G$.

An element $1 \neq x \in G$ is called torsion-free if $x^{\alpha} \neq 1$ for every $\alpha \in W$, $\alpha \neq 0$; otherwise it will be called periodic; the element $x \in G$ will be called proper if the lattice $L(<x>)$ is infinite.

The group G is torsion-free (proper) if all its elements are torsion-free (proper); it will be called nonperiodic if it contains both the proper and periodic elements; the set of all the periodic elements of G will be denoted by $t(G)$. It could be proved that $t(G)$ is W-ideal of G. It is clear that $G/t(G)$ is torsion-free A lattice isomorphism. $\varphi : L(G) \to L(G_1)$ is called normal if $[N(A)]^{\varphi} = N(A^{\varphi})$ for each subgroup $A \subseteq G$. The rank (G) is defined to be the rank $(G/t(G))$.

Proposition. *If the rank* $(G) \geq 1$ *then a lattice isomorphism* $\varphi : L(G) \to L(G_1)$ *is normal.*

Let $\Omega \subseteq W$ *be the group of units of W.*

Theorem. *Let* $f: L(G) \to L(G_1)$ *be a lattice isomorphism between nilpotent of class 2 W-power groups. If G contains a proper non-abelian W-subgroup, then there exists isomorphism* $\sigma: W \to W_1$ *and* σ*-semilinear isomorphism* $\varphi: G \to G_1$ *such that* $\varphi(A) = f(A)$ *for any W-subgroup* $A \subseteq G$ *and the mapping* $\varphi = [\varphi]^{\varepsilon}$ *will be either a semilinear isomorphism or a semilinear antiisomorphism with respect to the isomorphism* σ *for each* $\varepsilon \in \Omega$.

Remark. The following question arises naturally in connection with the theorem if every normal lattice isomorphism nilpotent of class 2 W-power group G is induced by an isomorphism of rank $G \geq 2$?

Below we give an example which answerly this question negatively.

Example. Let $G = \langle x_1, x_2 \rangle$ be W-power group over the principal ideal domain W, which is not a field with the defining relations

$$[x_1, x_2] = k,\ [k, x_1] = 1,\ [k, x_2] = 1,\ k^p = 1,$$

p is prime element of W, $p \neq 2$.

The elements x_1, x_2 are proper, i.e. $\exp(x_1) = \exp(x_2) = 1$

It is clear that $[G,G] = \langle k \rangle$, G is nilpotent of class 2 and each element l of G has the form

$$l = x_1^{\alpha_1} x_2^{\alpha_2} k^{\beta},$$

$\alpha_1, \alpha_2 \in W$, $\beta \notin \mathrm{id}(P) \leq W$ or $\beta = 0$.

Define one-to-one mapping $f : G \to G$ as follows

$$l' = f(l) = \begin{cases} l & if \quad \alpha_1\alpha_2 \in id(P), \\ l \cdot k^s & if \quad \alpha_1\alpha_2 \notin id(P), \\ s \notin id(P) & and \quad s + \alpha_1 + \alpha_2 \in id(P) \end{cases}$$

We can show that f induces a lattice automorphism of G and it is clear that f is neither a (semi) automorphism nor a (semi) antiautomorphism.

მათემატიკა

ნილპოტენტური კლასის 2 ჰოლის ხარისხოვანი W-ჯგუფების მესერული იზომორფიზმები

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