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Mathematics

Cyclic Configurations of Pentagon Linkages

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ABSTRACT. We investigate cyclic configurations of planar polygon and relate them to the topology of moduli space of the corresponding polygonal linkage. The main attention is given to quadrilateral and pentagon linkages. For a nondegenerate quadrilateral linkage, we prove that cyclic configurations are critical points of the signed area function on moduli space and their number is determined by the topology of moduli space. Moreover, cyclic configurations are critical points for the function defined as the product of lengths of two diagonals. For nondegenerate pentagon linkages, it is established that cyclic configurations are again critical points of the signed area function but the number of critical points is not determined by the topology of moduli space. *© 2008 Bull. Georg. Natl. Acad. Sci.*

Key words: *polygonal linkage, configuration space, cyclic polygon, signed area, generalized Heron polynomial.*

1. As usual, under a *cyclic polygon* we understand a polygon which can be inscribed in a circle, i.e., there exists a point (center of circumscribed circle) equidistant from all vertices of the polygon (see, e.g., [1]). Study of cyclic polygons has a long history starting with elementary classical results such as Ptolemy theorem and Brahmagupta formula (see, e.g., [1]). Important results on the existence and geometry of cyclic polygons were obtained by J. Steiner [1]. The study of cyclic polygons continues to attract considerable interest (see, e.g., [2]), in particular, due to the results and conjectures of D. Robbins concerned with computation of the areas of cyclic polygons [3]. Our results belong to the same circle of ideas but we use a different approach based on the concept of *configuration space of polygonal linkage* [4].

Mechanical linkages (or equivalently polygons with fixed lengths of the sides [1]) were actively studied from various points of view for at least 150 years (cf., e. g., [4]). In particular, configuration spaces of planar mechanical linkages were investigated in considerable detail [4], [5]. Those general results and approach developed by the third-named author in [6], [7] provided a natural background and main guidelines for our research. Our main aim is to show that cyclic configurations are critical points of the signed area function and that the number of cyclic configurations in many cases can be estimated and/or explicitly calculated without actually finding cyclic configurations. We believe that these conclusions are of interest since they confirm some of the general conjectures formulated in [7]. Our approach is based on a few paradigms of singularity theory, for which we refer to [8], and signature formulae for topological invariants [9]. It should be mentioned that quadrilateral and pentagon linkages from a similar viewpoint were studied in [5] and [6]. This paper can be considered as a sort of continuation of the research started in [5] and [6] but our setting and aims are essentially different.

2. Let us present first the necessary definitions and constructions. Recall that a k-gonal linkage L is defined by a k-tuple of nonnegative numbers l_i (called *sidelengths* of L) each of which is not greater than the sum of all other ones

[4]. The *N-th configuration space* $C_N(L)$ of such a linkage is defined as the collection of all k-tuples of points v_i in N-dimensional Euclidean space such that the distance between v_i and v_{i+1} is equal to l_i. Each such collection of points is called a *configuration* of L. Factoring this set by the natural diagonal action of SO(N) one obtains the *N-th moduli space* $M_N(L)$ [4]. Moduli spaces as well as configuration spaces are endowed with natural topologies induced by Euclidean metric. We will only consider the planar (second) moduli space $M_2(L)$ and denote it simply by $M(L)$.

It is easy to see that the moduli space can be identified with the subset of configurations such that $v_1 = (0,0)$, $v_2 = (l_1,0)$. It is also easy to realize that the moduli space is compact and can be represented as a level set of a certain quadratic mapping, which implies that, for generic values of l_i, the planar moduli space $M(L)$ has a natural structure of compact orientable manifold of dimension $k-3$. In fact, the condition of genericity needed in the last statement can be made quite precise. Let us say that linkage is *degenerate* if it has a configuration all vertices of which lie on the same straight line. A minute's thought shows that this happens if and only if there exists a k-tuple of "signs" $s_i = \pm 1$ such that $\Sigma\, s_i l_i = 0$. Now, it is possible to show that moduli space $M(L)$ is smooth (does not have singular points) if and only if linkage L is nondegenerate (see, e. g., [4]). For smooth moduli space, a natural idea is to investigate its topology using Morse theory of some natural smooth function on it. We will use this idea for two natural functions on configuration space.

To this end recall that, for any configuration V of L with vertices $v_i = (x_i, y_i)$, its *signed area* $A(V)$ is defined by putting $A(V) = (x_1y_2 - x_2y_1) + \ldots + (x_ny_1 - x_1y_n)$. Obviously, this formula defines a continuous function on moduli space $M(L)$. If a configuration space is smooth then we obtain a smooth function A on compact manifold $M(L)$ and can consider its critical points. In fact, it is possible to show that A defines a Morse function on each smooth moduli space [7] and so one can use Morse theory to study the topology of moduli spaces (see [7]). We were able to show that, for $k = 4$ and $k = 5$, all critical points of A in $M(L)$ are given by cyclic configurations, which can be considered as the main result of this note. To find critical points of A we use the classical Lagrange method. For $k = 4$, the first-named author proved the desired result by direct computation and as a by-product obtained that the number of cyclic configurations is equal to two times the number of components of moduli space. For $k = 5$, the necessary computations appeared too hard to do by hand but the second-named author managed to use symbolic computations to obtain the result. Moreover, the third-named author noticed that similar conclusions hold for several other natural functions on moduli space of a nondegenerate quadrilateral. One such function D on $M(L)$ is given by the product of lengths of the two diagonals and its critical points also coincide with cyclic configurations. Further functions with such property may be obtained from physical considerations (cf. [7]). In our opinion, these results reveal new aspects of cyclic configurations and suggest interesting problems some of which are mentioned in the sequel.

3. We begin by indicating polynomial systems defining the moduli space and cyclic configurations of a given linkage. For a quadrilateral linkage Q, the moduli space can be naturally identified with the set of all real solutions to the following system of three equations in four unknowns:

$$(x - l_1)^2 + y^2 = (l_2)^2\ ,\ (x - s)^2 + (y - t)^2 = (l_3)^2,\ s^2 + t^2 = (l_4)^2,$$

where $v_3 = (x, y)$, $v_4 = (s, t)$ are the "movable" vertices of configuration. Obviously, a configuration is cyclic if and only if there exist three real numbers p, q, r satisfying the system:

$$p^2 + q^2 = r^2,\ (p - l_1)^2 + q^2 = r^2,\ (p - x)^2 + (q - y)^2 = r^2,\ (p - u)^2 + (q - v)^2 = r^2\ .$$

Joining both groups of equations we obtain a system of 7 quadratic equations in 7 unknowns such that the first four coordinates (x, y, s, t) of its real solutions give the cyclic configurations of Q. A direct computation shows that the Jacobian of this system does not vanish identically and so, for *generic* values of l_i, the set of real solutions to this system is finite. Restriction to generic values of sidelengths is necessary since each rhomboid ($l_1 = l_2 = l_3 = l_4$) has a continual set of cyclic configurations (e. g., all configurations with $v_4 = v_2 = (l_1, 0)$ are obviously cyclic).

Thus, for generic sidelengths, one can investigate this system by standard methods of singularity theory. In particular, one can compute the bifurcation diagram [8] of this system and find the number of components (chambers) of its complement. Next, one chooses a point in each component and finds the number of real solutions using the signature formula for Euler characteristic [9]. Since by Ehresmann theorem the number of solutions is constant on each chamber (see, e. g., [8]), in this way we obtain all possible values of the number of cyclic configurations for nondegenerate quadrilateral linkages. This strategy is in principle applicable for a linkage with an arbitrary number of sides but it requires a lot of computational work. In the case of quadrilateral, the same aim can be achieved more directly. We present now a complete answer for $k = 4$ together with an (outline of) elementary proof found by the first-named author.

Theorem 1. *Let Q be a nondegenerate quadrilateral linkage. Then all critical points of the signed area function A on planar moduli space $M(Q)$ are given by cyclic configurations. The number of cyclic configurations of Q can be two or four. The number of cyclic configurations is two if and only if the sum of lengths of the biggest and smallest sidelengths of Q is greater than the sum of lengths of the two other sides. Moreover, the number of cyclic configurations is equal to the two times the number of components of moduli space $M(Q)$.*

Proof. Let the sidelengths of Q be a, b, c, d and let α and β be the oriented angles between consecutive sides of Q at vertices v_2 and v_4, respectively, which will be used as generalized coordinates on $M(Q)$. Computing length of the diagonal joining v_1 and v_3 by cosine theorem in two ways, we find that the equation of $M(L)$ is: $a^2 + b^2 - c^2 - d^2 - 2\,a\,b \cos\alpha + 2\,c\,d \cos\beta = 0$. We can now apply Lagrange method to the function $A = a\,b \sin\alpha + c\,d \sin\beta$ with respect to these generalized coordinates. Computing gradients of A and of the above equation we obtain that they are equal to $(a\,b \cos\alpha, c\,d \cos\beta)$ and $(2\,a\,b \sin\alpha, -2\,c\,d \sin\beta)$, respectively. Writing down the condition of linear dependence of the two gradients at a critical point, we easily find out that it is equivalent to $\sin(\alpha + \beta) = 0$ or $\alpha + \beta = k\,\pi$. In our situation k can only have values 1 or 0. If $k = 1$ we obviously get a convex cyclic configuration, while for $k = 0$ one obtains a self-intersecting cyclic configuration of L. Notice now that in the first case we have $\cos\alpha = -\cos\beta$ while in the second case $\cos\alpha = \cos\beta$. Eliminating $\cos\beta$ from the above equation we find that in both cases it is possible to express $\cos\alpha$ through the lengths a, b, c, d. This already shows that the number of cyclic configurations can be either two or four. Next, for $k = 1$ we get $\cos\alpha = (a^2 + b^2 - c^2 - d^2) / 2(a\,b + c\,d)$. It is easy to see that if the biggest of those numbers does not exceed the sum of others, the absolute value of the fraction in the right-hand side does not exceed one. Hence a convex cyclic configuration exists for any quadrilateral Q. In the second case we obtain a similar fraction but with denominator equal to $2(a\,b - c\,d)$. Solving an inequality expressing the fact that the absolute value of the right-hand side should not exceed one, we easily arrive at the third statement given in the theorem. Finally, comparing the latter condition with the known results on the topology of moduli spaces of quadrilaterals [5], one obtains the last statement and completes the proof.

This result combined with the nondegeneracy property of A established in [7] shows that A is a perfect Morse function on $M(Q)$. It also enables one to find critical values of A on $M(L)$ by Brahmagupta-Robbins formulæ [3].

Corollary 1. *Critical values of A can be found as the real roots of the following two equations:*

$$16\,x^2 = 2\,a^2 b^2 + \ldots + 2\,c^2 d^2 - a^4 - b^4 - c^4 - d^4 \pm 8\,a\,b\,c\,d\,.$$

4. For pentagon linkages the problem is more complicated and results are less complete.

Theorem 2. *Let P be a nondegenerate pentagon linkage. Then all critical points of the signed area function A on the planar moduli space $M(P)$ are given by cyclic configurations. The number of cyclic configurations of P can take values 4, 6, 8, 10, 12, 14.*

The first statement is obtained by introducing angular coordinates and rather lengthy analysis of Lagrange equations with the aid of computer algebra. The second statement follows from the upper estimate established by D. Robbins [3] and explicit examples of pentagons for which A has the indicated amounts of critical points. In this case it appears impossible to determine the number of cyclic configurations from the topology of $M(P)$ because there exist pentagons P, P' such that $M(P)$ is homeomorphic to $M(P')$ but P and P' have different numbers of cyclic configurations. Examples of such pentagons were found by the second-named author by means of computer experiments using the algorithms for calculation of the local degree. This implies, in particular, that A is not always a perfect Morse function on moduli spaces of pentagon linkages and we are led to the problem of characterizing those sidelengths for which A is a perfect Morse function on $M(P)$. We also obtain an analog of Corollary 1.

Corollary 2. *Critical values of A coincide with the real roots of the generalized Heron polynomial found by D. Robbins.*

An interesting problem is to work out an algorithm for finding the number of cyclic configurations in terms of the sidelengths.

5. Consider again a quadrilateral linkage Q and define a function $D : M(Q) \to \mathbf{R}$ as follows: for $V \in M(Q)$ let $D(V)$ be equal to the product of lengths of diagonals of V.

Theorem 3. *For a nondegenerate quadrilateral Q, the critical points of D are given by cyclic configurations of Q. The absolute maximum and minimum of D are equal to $\pm(a\,c + b\,d)$, respectively.*

The proof can be again obtained using Lagrange method. The statement about extrema follows from Ptolemy theorem [1]. Similar results can be obtained for several other functions on moduli space arising as energy functions of certain physical systems associated with linkage L. For example it is easy to see that the problem of minimizing electrostatic energy of four equal charges placed at vertices of L reduces to finding extrema of the function $(d_{13})^{-2} + (d_{24})^{-2}$, where d_{ij} are the distances between the ith and jth vertices. The third-named author was able to show that these extrema are again given by cyclic configurations of L and their values can be explicitly calculated in terms of sidelengths.

Generalizations of such kind and full proofs of the above results will be presented in a forthcoming detailed publication.

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მათემატიკა

ხუთსახსრულიანი მექანიზმების წრეწირში ჩახაზვადი კონფიგურაციები

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შესწავლილია ბრტყელი ხუთკუთხედების წრეწირში ჩახაზვადი კონფიგურაციები და მათი კავშირი მოდულების სივრცის ტოპოლოგიასთან. ოთხკუთხედის შემთხვევაში დამტკიცებულია, რომ წრეწირში ჩახაზვადი კონფიგურაციები ორიენტირებული ფართობის კრიტიკული წერტილებია და მათი რაოდენობა განისაზღვრება მოდულების სივრცის ტოპოლოგიით. ასეთივე შედეგი მიღებულია ფუნქციისთვის, რომელიც განიმარტება როგორც ოთხკუთხედის დიაგონალთა სიგრძეების ნამრავლი. ანალოგიური შედეგი სამართლიანია ხუთკუთხედებისათვის, მაგრამ ამ შემთხვევაში წრეწირში ჩახაზვადი კონფიგურაციების რაოდენობა მოდულების სივრცის ტოპოლოგიით უკვე არ განისაზღვრება.

REFERENCES

1. *H. Coxeter, S. Greitzer* (1967), Geometry Revisited, Washington.
2. *M. Fedorchuk, I. Pak* (2005), Duke Math. J., **129**, 2: 371–404.
3. *D. Robbins* (1994), Discrete Comput. Geom., **12**: 223–236.
4. *R. Connelly, E. Demaine* (2004), Handbook of discrete and computational geometry, 2nd ed. CRC Press, Boca Raton: 197–218.
5. *C. Gibson, P. Newstead* (1986), Acta Applic. Math., **7**:113–135.
6. *Г. Химшиашвили* (2002), Зап. науч. семинаров ПОМИ, **292**: 120–129.
7. *G. Khimshiashvili* (2008), J. Math. Sci., **59**: 131-140.
8. *В. И. Арнольд, А. Н. Варченко, С. М. Гусейн-Заде* (2005), Особенности дифференцируемых отображений, Москва.
9. *G. Khimshiashvili* (2001), A. Razmadze Math. Inst. Proc., **125**:1–121.

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