Mathematics

Martingale Measures for the Geometrical Gaussian Martingale

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ABSTRACT. The problems of finding of the martingale measures for some class of stochastic processes in discrete time are investigated. A special class of densities is introduced and minimal relative entropy martingale measure for the process represented by exponential Gaussian martingale is constructed. © 2008 Bull. Georg. Natl. Acad. Sci.

Key words: Gaussian martingale, entropy, martingale measure.

On the filtered probability space \((\Omega, F, (F_n)_{0 \leq n \leq N}, P)\) consider the stochastic process of discrete time with real values

\[ S_n = S_0 \exp \{M_n\}, \quad n = 1, \ldots, N, \tag{1} \]

where \(S_0 > 0\) is deterministic, \((M_n, F_n)_{0 \leq n \leq N}, M_0 = 0,\) is a Gaussian martingale with quadratic characteristic \(\langle M \rangle_n = EM_n^2\). We describe the evolution of a risky asset by this scheme.

Now we construct the density process \((Z_n, n = 0, 1, \ldots, N). Define

\[ Z_n^\psi = Z_n^\psi \frac{\exp \{\psi_n\}}{E[\exp \{\psi_n\} / F_{n-1}]} , \quad Z_0 = 1, \tag{2} \]

where \((\psi_n, F_n)_{0 \leq n \leq N}\) is some stochastic sequence. It is clear that \(Z_n^\psi\) is \(F_n\) measurable.

Let

\[ E \exp \{\psi_n\} < \infty, \text{ then } (Z_n^\psi, F_n)_{0 \leq n \leq N} \text{ is } P \text{ martingale.} \]

Consider the measure

\[ Q_\psi (A) = \int_A Z_n^\psi (\omega) dP(\omega), A \in F, \tag{3} \]

which is equivalent to \(P\).

If \(\psi = (\psi_n, F_n)\) satisfies

\[ E[\exp \{\psi_n + \Delta M_n\} / F_{n-1}] = E[\exp \{\psi_n\} / F_{n-1}](P - a.s), \tag{4} \]

then \(Q_\psi\) is a martingale measure for \(S\), i.e. \(Q_\psi\) is equivalent to \(P\) and \(S\) is \((F_n, Q_\psi)\) martingale.

Really, from (1),(2) we have...
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\[
E^{Q^{\psi}}[S_n/F_{n-1}] = \frac{E[S_nZ_n^{\psi}/F_{n-1}]}{E[Z_n^{\psi}/F_{n-1}]} = \frac{S_{n-1}Z_{n-1}^{\psi}}{Z_{n-1}^{\psi}} E[\exp\{\psi_n + \Delta M_n\}/F_{n-1}] \]

and using (4) we have \( E^{Q^{\psi}}(S_n/F_{n-1}) = S_{n-1} \).

Consider the case when \( \psi = (\psi_n, F_n) \) is the sequence of discrete stochastic integrals

\[
\psi_n = \sum_{k=1}^{n} \phi(k)\Delta M_k,
\]

with predictable \( \phi(k) \), i.e. \( \phi(k) \) is \( F_{k-1} \) measurable. Then from (2), (4) the measure \( Q^* \) with density

\[
Z_n = Z_{n-1} \frac{\exp\{-M_n/2\}}{E[\exp\{-M_n/2\}/F_{n-1}]} \]

is the martingale measure for \( S \), which is \((P-a.s)\) unique among equivalent to \( P \) measures with density process

\[
Z_n = Z_{n-1} \frac{\exp\{\sum_{k=1}^{n} \phi(k)\Delta M_k\}}{E[\exp\{\sum_{k=1}^{n} \phi(k)\Delta M_k\}/F_{n-1}]}.
\]

This density coincides with the density obtained by conditional Escher transform

\[
Z_n = Z_{n-1} \frac{\exp\{cM_n\}}{E[\exp\{cM_n\}/F_{n-1}]} ,
\]

where \( c \) is some constant (see [2], p.540).

Let \( \psi_n = a\Delta M_n^2 + b\Delta M_n \), where \( a \) and \( b \) are constants, \( \psi_n \) is \( F_n \) measurable for each \( n \). In this case the condition (4) has the form

\[
E \exp\{a\Delta M_n^2 + (b+1)\Delta M_n\} = E \exp\{a\Delta M_n^2 + b\Delta M\},
\]

which is fulfilled if \( b = -\frac{1}{2} \) and for any constant \( a \) the class of martingale measures for \( S \) is defined by the density process

\[
\tilde{Z}_n = \tilde{Z}_{n-1} \frac{\exp\{a(\Delta M_n)^2 - \Delta M_n\}}{E \exp\{a(\Delta M_n)^2 - \Delta M_n\}} = \prod_{k=1}^{n} \frac{\exp\{a(\Delta M_k)^2 - \Delta M_k\}}{E \exp\{a(\Delta M_k)^2 - \Delta M_k\}} ,
\]

(5)

Let us consider the class of martingale measures \( Q^a \) \((a \in R)\) with density (5), i.e.

\[
\frac{dQ_a}{dP} = \tilde{Z}_N = \prod_{k=1}^{N} \frac{\exp\{a(\Delta M_k)^2 - \Delta M_k\}}{E \exp\{a(\Delta M_k)^2 - \Delta M_k\}} .
\]

(6)

Now we find the constant \( a^* \) and corresponding probability measure \( Q_{a^*} \) which minimizes the relative entropy. Recall that the relative entropy of probability measure \( Q \) with respect to probability measure \( P \) is defined as

\[
I(Q, P) = \begin{cases} E_P \left[ \frac{dQ}{dP} \ln \frac{dQ}{dP} \right] , & \text{if } Q << P , \\
\infty , & \text{otherwise.}
\end{cases}
\]

So we have to find the constant \( a^* \) and corresponding measure \( Q_{a^*} \) with density (6) for which

\[
I(Q_{a^*}, P) \rightarrow \min .
\]

Let us computer \( E_P[\tilde{Z}_N \ln \tilde{Z}_N] \). In the beginning consider
\[ A_k = E \exp \{a(\Delta M_k)^2 - \Delta M_k\} = \frac{1}{\sqrt{2\pi \Delta(\langle M \rangle_k)}} \int_R e^{ax - \frac{1}{2}x^2} e^{-\frac{x^2}{2\Delta(\langle M \rangle_k)}} dx = \frac{e^{\frac{\Delta(\langle M \rangle_k)}{2\Delta(\langle M \rangle_k)k}}}{\sqrt{1 - 2a\Delta(\langle M \rangle_k)}}, \quad (7) \]

where \( a < \frac{1}{2\Delta(\langle M \rangle_k)} \). So, we have that

\[ \tilde{Z}_N \ln \tilde{Z}_N = \sum_{k=1}^{N} [(a(\Delta M_k)^2 - \Delta M_k) - \frac{\Delta M_k}{2}] A_k / \prod_{k=1}^{N} A_k - \ln A_k \]

Consider the mathematical expectation

\[ E_P[\tilde{Z}_N \ln \tilde{Z}_N] = \sum_{k=1}^{N} \left[ E \frac{\exp \{a(\Delta M_k)^2 - \Delta M_k\} - \frac{\Delta M_k}{2}}{A_k} - \ln A_k \right], \quad (8) \]

using (7) and (8) we obtain for the relative entropy the following representation

\[ I(Q, P) = \sum_{k=1}^{N} \left[ a(\Delta(\langle M \rangle_k) + \Delta(\langle M \rangle_k)/2 \right] + \ln \sqrt{1 - 2a\Delta(\langle M \rangle_k)}. \]

Now we find the parameter \( a \) for which the \( I(Q, P) \) takes minimum. Consider the derivative

\[ \frac{dI(Q, P)}{da} = -8a^2 \sum_{k=1}^{N} \frac{\Delta(\langle M \rangle_k)^3}{2(1 - 2a\Delta(\langle M \rangle_k))^3} + 4a \sum_{k=1}^{N} \frac{\Delta(\langle M \rangle_k)^2}{2(1 - 2a\Delta(\langle M \rangle_k))^3} + \sum_{k=1}^{N} \frac{\Delta(\langle M \rangle_k)^2}{2(1 - 2a\Delta(\langle M \rangle_k))^3}. \quad (9) \]

Assume that \( \Delta(\langle M \rangle_k) = 1 \), i.e. \( \langle M \rangle_k = k \), then from (9) we obtain the quadratic equation for \( a \)

\[ 8a^2 - 4a - 1 = 0 \]

Solutions of this equation are \( a_1 = \frac{1 + \sqrt{3}}{4} \) and \( a_2 = \frac{1 - \sqrt{3}}{4} \). Recall that \( a < \frac{1}{2\Delta(\langle M \rangle_k)} \), if \( \Delta(\langle M \rangle_k) = 1 \), then \( a < \frac{1}{2} \) and the constant \( a^* = \frac{1 - \sqrt{3}}{4} \). This is the point of minimum, because \( \frac{d^2I(Q, P)}{da^2} > 0 \) in \( a^* \).

So, we have proved the following

**Theorem.** Let \( S_n = S_0 \exp \{M_n\} \), \( n = 1, ..., N, S_0 > 0 \), where \( (M_n, F_n)_{0 \leq n \leq N} \), \( M_0 = 0 \) is the Gaussian martingale with quadratic characteristic \( \langle M \rangle_n = EM_n^2 \). In the class of martingale measures with densities defined by (6) the minimal relative entropy martingale measure has the density

\[ \frac{dQ_n}{dP} = \tilde{Z}_N \prod_{k=1}^{N} E \exp \left[ \frac{1 - \sqrt{3}}{4} (\Delta M_k)^2 - \frac{\Delta M_k}{2} \right]. \]

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