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*Mathematics*

# Martingale Measures for the Geometrical Gaussian Martingale

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**ABSTRACT. The problems of finding of the martingale measures for some class of stochastic processes in discrete time are investigated. A special class of densities is introduced and minimal relative entropy martingale measure for the process represented by exponential Gaussian martingale is constructed.** *© 2008 Bull. Georg. Natl. Acad. Sci.*

***Key words:*** *Gaussian martingale, entropy, martingale measure.*

On the filtered probability space $(\Omega, F, (F_n)_{0\le n\le N}, P)$ consider the stochastic process of discrete time with real values

$$S_n = S_0 \exp\{M_n\},\ \ n = 1,...,N\ , \tag{1}$$

where $S_0 > 0$ is deterministic, $(M_n, F_n)_{0\le n\le N}, M_0 = 0$, is a Gaussian martingale with quadratic characteristic $\langle M\rangle_n = EM_n^2$. We describe the evolution of a risky asset by this scheme.

Now we construct the density process $Z_n, n = 0,1,...,N$. Define

$$Z_n^{\psi} = Z_{n-1}^{\psi} \frac{\exp\{\psi_n\}}{E[\exp\{\psi_n\}/F_{n-1}]},\ \ Z_0 = 1, \tag{2}$$

where $(\psi_n, F_n)_{0\le n\le N}$ is some stochastic sequence. It is clear that $Z_n^{\psi}$ is $F_n$ measurable.

Let

$$E\exp\{\psi_n\} < \infty,\ \text{then}\ (Z_N^{\psi}, F_n)_{0\le n\le N}\ \text{is}\ P\ \text{martingale.}$$

Consider the measure

$$Q_{\psi}(A) = \int_A Z_N^{\psi}(\omega)dP(\omega), A \in F, \tag{3}$$

which is equivalent to $P$.

If $\psi = (\psi_n, F_n)$ satisfies

$$E[\exp\{\psi_n + \Delta M_n\}/F_{n-1}] = E[\exp\{\psi_n\}/F_{n-1}], (P - a.s), \tag{4}$$

then $Q_{\psi}$ is a martingale measure for $S$, i.e. $Q_{\psi}$ is equivalent to $P$ and $S$ is $(F_n, Q_{\psi})$ martingale.

Really, from (1),(2) we have

$$E^{Q_\psi}[S_n / F_{n-1}] = \frac{E[S_n Z_N^\psi / F_{n-1}]}{E[Z_N^\psi / F_{n-1}]} = \frac{S_{n-1} Z_{n-1}^\psi}{Z_{n-1}^\psi} \frac{E[\exp\{\psi_n + \Delta M_n\} / F_{n-1}]}{E[\exp\{\psi_n\} / F_{n-1}]}$$

and using (4) we have $E^{Q\psi}(S_n / F_{n-1}) = S_{n-1}$.

Consider the case when $\psi = (\psi_n, F_n)$ is the sequence of discrete stochastic integrals

$$\psi_n = \sum_{k=1}^{n} \varphi(k) \Delta M_k ,$$

with predictable $\varphi(k)$, i.e. $\varphi(k)$ is $F_{k-1}$ measurable. Then from (2), (4) the measure $Q^*$ with density

$$Z_n^* = Z_{n-1}^* \frac{\exp\{-\frac{M_n}{2}\}}{E[\exp\{-\frac{M_n}{2}\} / F_{n-1}]} = Z_{n-1} \exp\{-\frac{\Delta M_n}{2} + \frac{\Delta \langle M \rangle_n}{4}\} = \exp\{-\frac{M_n}{2} + \frac{\langle M \rangle_n}{4}\}, Z_0 = 1,$$

is the martingale measure for $S$, which is $(P-a.s)$ unique among equivalent to $P$ measures with density process

$$Z_n = Z_{n-1} \frac{\exp\{\sum_{k=1}^{n} \varphi(k) \Delta M_k\}}{E[\exp\{\sum_{k=1}^{n} \varphi(k) \Delta M_k\} / F_{n-1}]} .$$

This density coincides with the density obtained by conditional Escher transform

$$Z_n = Z_{n-1} \frac{\exp\{c M_n\}}{E[\exp\{c M_n\} / F_{n-1}]},$$

where $c$ is some constant (see [2], p.540).

Let $\psi_n = a \Delta M_n^2 + b \Delta M_n$, where $a$ and $b$ are constants, $\psi_n$ is $F_n$ measurable for each $n$. In this case the condition (4) has the form

$$E \exp\{a \Delta M_n^2 + (b+1) \Delta M\} = E \exp\{a \Delta M_n^2 + b \Delta M\},$$

which is fulfilled if $b = -\frac{1}{2}$ and for any constant $a$ the class of martingale measures for $S$ is defined by the density process

$$\widetilde{Z}_n = \widetilde{Z}_{n-1} \frac{\exp\{a(\Delta M_n)^2 - \frac{\Delta M_n}{2}\}}{E \exp\{a(\Delta M_n)^2 - \frac{\Delta M_n}{2}\}} = \prod_{k=1}^{n} \frac{\exp\{a(\Delta M_k)^2 - \frac{\Delta M_k}{2}\}}{E \exp\{a(\Delta M_k)^2 - \frac{\Delta M_k}{2}\}}, \tag{5}$$

Let us consider the class of martingale measures $Q^a$ $(a \in R)$ with density (5), i.e.

$$\frac{dQ_a}{dP} = \widetilde{Z}_N = \prod_{k=1}^{N} \frac{\exp\{a(\Delta M_k)^2 - \frac{\Delta M_k}{2}\}}{E \exp\{a(\Delta M_k)^2 - \frac{\Delta M_k}{2}\}} . \tag{6}$$

Now we find the constant $a^*$ and corresponding probability measure $Q_a^*$ which minimizes the relative entropy. Recall that the relative entropy of probability measure $Q$ with respect to probability measure $P$ is defined as

$$I(Q,P) = \begin{cases} E_P[\frac{dQ}{dP} \ln \frac{dQ}{dP}], & if\ Q << P, \\ \infty, & otherwise. \end{cases}$$

So we have to find the constant $a^*$ and corresponding measure $Q_a^*$ with density (6) for which

$$I(Q_a^*, P) \to \min .$$

Let us compute $E_P[\widetilde{Z}_N \ln \widetilde{Z}_N]$. In the beginning consider

$$A_k = E\exp\{a(\Delta M_k)^2 - \frac{\Delta M_k}{2}\} = \frac{1}{\sqrt{2\pi\Delta\langle M\rangle_k}}\int_R e^{ax^2-\frac{1}{2}x} e^{-\frac{x^2}{2\Delta\langle M\rangle_k}}dx = \frac{e^{\frac{\Delta\langle M\rangle_k}{8(1-2a\Delta\langle M\rangle_k)}}}{\sqrt{1-2a\Delta\langle M\rangle_k}}, \quad (7)$$

where $a < \frac{1}{2\Delta\langle M\rangle_k}$. So, we have that

$$\widetilde{Z}_N \ln \widetilde{Z}_N = \sum_{k=1}^{N}[(a(\Delta M_k)^2 - \frac{\Delta M_k}{2})\prod_{k=1}^{N}\frac{\exp\{a(\Delta M_k)^2 - \frac{\Delta M_k}{2}\}}{A_k} - \ln A_k \prod_{k=1}^{N}\frac{\exp\{a(\Delta M_k)^2 - \frac{\Delta M_k}{2}\}}{A_k}]$$

Consider the mathematical expectation

$$E_P[\widetilde{Z}_N \ln \widetilde{Z}_N] = \sum_{k=1}^{N}[E\frac{\exp\{a(\Delta M_k)^2 - \frac{\Delta M_k}{2}\}(a(\Delta M_k)^2 - \frac{\Delta M_k}{2})}{A_k} - \ln A_k], \quad (8)$$

using (7) and (8 ) we obtain for the relative entropy the following representation

$$I(Q_a, P) = \sum_{k=1}^{N}[\frac{a\Delta\langle M\rangle_k}{(1-2a\Delta\langle M\rangle_k)} + \frac{\Delta\langle M\rangle_k}{8(1-2a\Delta\langle M\rangle_k)^2} + \ln\sqrt{1-2a\Delta\langle M\rangle_k}].$$

Now we find the parameter $a$ for which the $I(Q_a, P)$ takes minimum. Consider the derivative

$$\frac{dI(Q_a,P)}{da} = -8a^2\sum_{k=1}^{N}\frac{\Delta\langle M\rangle_k^{\ 3}}{2(1-2a\Delta\langle M\rangle_k)^3} + 4a\sum_{k=1}^{N}\frac{\Delta\langle M\rangle_k^{\ 2}}{2(1-2a\Delta\langle M\rangle_k)^3} + \sum_{k=1}^{N}\frac{\Delta\langle M\rangle_k^{\ 2}}{2(1-2a\Delta\langle M\rangle_k)^3}. \quad (9)$$

Assume that $\Delta\langle M\rangle_k = 1$, i.e. $\langle M\rangle_k = k$, then from (9) we obtain the quadratic equation for $a$

$$8a^2 - 4a - 1 = 0$$

Solutions of this equation are $a_1 = \frac{1+\sqrt{3}}{4}$ and $a_2 = \frac{1-\sqrt{3}}{4}$. Recall that $a < \frac{1}{2\Delta\langle M\rangle_k}$, if $\Delta\langle M\rangle_k = 1$, then $a < \frac{1}{2}$ and the constant $a^* = \frac{1-\sqrt{3}}{4}$. This is the point of minimum, because $\frac{d^2 I(Q_a,P)}{da^2} > 0$ in $a^*$.

So, we have proved the following

**Theorem**. *Let* $S_n = S_0\exp\{M_n\}, n = 1,...,N, S_0 > 0,$ *where* $(M_n, F_n)_{0\le n\le N}, M_0 = 0$ *is the Gaussian martingale with quadratic characteristic* $\langle M\rangle_n = EM_n^2$. *In the class of martingale measures with densities defined by (6) the minimal relative entropy martingale measure has the density*

$$\frac{dQ_{a^*}}{dP} = \widetilde{Z}_N = \prod_{k=1}^{N}\frac{\exp\{\frac{1-\sqrt{3}}{4}(\Delta M_k)^2 - \frac{\Delta M_k}{2}\}}{E\exp\{\frac{1-\sqrt{3}}{4}(\Delta M_k)^2 - \frac{\Delta M_k}{2}\}}.$$

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*მათემატიკა*

# მარტინგალური ზომები გეომეტრიული გაუსის მარტინგალისათვის

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