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Mathematics

Boundedness in Lebesgue Spaces with Variable Exponent of the Calderon Singular Operator on Carleson Curves

Vakhtang Kokilashvili

Academy Member, A. Razmadze Mathematical Institute, Tbilisi

ABSTRACT. We prove the boundedness of the Calderon singular integral operator in variable exponent weighted Lebesgue spaces $L^{p(\cdot)}(\Gamma, w)$ on arbitrary Carleson curve under the assumption that $p(t)$ satisfies the log-condition on Γ. *© 2008 Bull. Georg. Natl. Acad. Sci.*

Key words: *variable exponent weighted Lebesgue space, Carleson curve, the Calderon singular integral.*

1. Introduction

Let $\Gamma = \{t \in C : t = t(s), \quad 0 \le s \le l < \infty\}$ be a simple rectifiable curve with arc-length measure. In the sequel we denote

$$\gamma(t,r) = \Gamma \cap B(t,r), \; t \in \Gamma, \; r > 0\,, \tag{1.1}$$

where $B(t,r) = \{z \in C : |z - t| < r\}$. We also denote by $|\gamma(t,r)|$ arc-length measure of $\gamma(r,t)$.

We remind that a curve is called Carleson curve (regular curve), if there exists a constant $c_0 > 0$ not depending on *t* and *r*, such that

$$|\gamma(t,r)| \le c_0 r\,. \tag{1.2}$$

We consider the Calderon singular integral operator

$$C_\Gamma(a,f) = \int_\Gamma \frac{a(\tau) - a(t)}{(\tau - t)^2} f(\tau) d\tau \tag{1.3}$$

on Carleson curves Γ and establish that C_Γ is bounded in weighted spaces $L^{p(\cdot)}(\Gamma, w)$, $w(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k}$, $t_k \in \Gamma$ with variable exponent $p(t)$ (see definitions in Section 2), under the assumption that $p(t)$ satisfies the standard log-condition.

2. Definitions

Let p be a measurable function on Γ such that $p : \Gamma \to (1, \infty)$. In what follows we assume that p satisfies the conditions

$$1 < p_- := \operatorname*{ess\,inf}_{t \in \Gamma} p(t) \le \operatorname*{ess\,sup}_{t \in \Gamma} p(t) =: p_+ < \infty\,, \tag{2.1}$$

$$|p(t)-p(\tau)| \le \frac{A}{\ln|t-\tau|}, \quad t\in\Gamma, \quad \tau\in\Gamma, \quad |t-\tau|\le\frac{1}{2}. \tag{2.2}$$

In the case where Γ is an infinite curve, we also assume that p satisfies the following condition at infinity

$$|p(t)-p(\tau)| \le \frac{A_\infty}{\ln\left|\frac{1}{t}-\frac{1}{\tau}\right|}, \qquad \left|\frac{1}{t}-\frac{1}{\tau}\right|\le\frac{1}{2}, \quad |t|\ge L, \qquad |\tau|\ge L \tag{2.3}$$

for some $L>0$.

From (2.3) it follows that there exists $p_\infty = \lim\limits_{|t|\to\infty} p(t)$ and $|p(t)-p(\infty)| \le \frac{A_\infty}{\ln|t|}$, $|t|\ge\max\{L,2\}$.

The conditions (2.2), (2.3) are called the log-conditions.

The generalized Lebesgue space with variable exponent is defined via the modular

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda>0:\ I_\Gamma^p\left(\frac{f}{\lambda}\right)\le 1\right\}.$$

by the norm

$$I_\Gamma^p(f) := \int_\Gamma |f(t)|^{p(t)} ds$$

By $L^{p(\cdot)}(\Gamma,w)$ we denote the weighted Banach space of all measurable functions $f:\Gamma\to C$ such that

$$\|f\|_{p(\cdot),w} = \|wf\|_{p(\cdot)} < \infty.$$

We denote $p'(t)=\frac{p(t)}{p(t)-1}$.

3. The main statements

In the sequel we consider the power weights of the form

$$w(t)=\prod_{k=1}^{n}|t-t_k|^{\beta_k}, \ t_k\in\Gamma, \quad t_i\ne t_j \ \text{ when } i\ne j. \tag{3.1}$$

Theorem 1. *Let*

i) Γ be a simple Carleson curve with finite or infinite length;

ii) the functions p(t) and r(t) satisfy conditions (2.1) and (2.2) in the case of finite Γ and also (2.3) when Γ is infinite;

iii) $a'\in L^{r(\cdot)}(\Gamma)$.

Then the operator $C_\Gamma(a,\cdot)$ is bounded from $L^{p(\cdot)}(\Gamma)$ into $L^{q(\cdot)}(\Gamma)$ where

$$\frac{1}{q(t)}=\frac{1}{p(t)}+\frac{1}{r(t)}.$$

Theorem 2. *Let*

i) Γ and p satisfy conditions from Theorem 1.

ii) $a'\in L^\infty(\Gamma)$.

Then the operator $C_\Gamma(a,\cdot)$ is bounded in the space $L_w^{p(\cdot)}(\Gamma)$ with power weight w of the form (3.1) if

$$-\frac{1}{p(t_k)}<\beta_k<\frac{1}{p'(t_k)}, \qquad k=1,2,\ldots,n$$

and also

$$-\frac{1}{p(\infty)}<\beta+\sum_{k=1}^{n}\beta_k<\frac{1}{p'(\infty)}$$

in the case Γ is infinite.

Theorem 3. *Let*

i) Γ be a simple closed rectifiable curve;

ii) p(t) satisfy conditions (2.1) and (2.2);

iii) there exist positive constants m and M such that

$$0 < m \le |a'(t)| \le M < \infty .$$

Then from the boundedness of C_Γ *in* $L^{p(\cdot)}(\Gamma)$ *follows that* Γ *is a Carleson curve.*

For the case $\Gamma = R^1$, *p*=const. and $a' \in L^\infty(\Gamma)$ we refer to [1]. When Γ is a Carleson curve, *p*=const. and $a(t) \equiv t$ Theorem 2 is due to G. David [2] and when an exponent is variable see [3].

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ვ. კოკილაშვილი

აკადემიის წევრი, ა. რაზმაძის მათემატიკის ინსტიტუტი, თბილისი

ნაშრომში მოყვანილია წონიანი უტოლობები კარლესონის წირებზე განსაზღვრული კალდერონის სინგულარული ოპერატორებისათვის $L^{p(\cdot)}(\Gamma, w)$ სივრცეებში, როცა $p(t)$ ფუნქცია აკმაყოფილებს ე.წ. ლოგარითმულ პირობას.

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