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Mathematics

On an Integral Square Deviation Measure with the Weight of "Delta-Functions" of the Rosenblatt-Parzen Probability Density Estimator

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ABSTRACT. The limit distribution of an integral square deviation with the weight of "delta-functions" of the Rosenblatt-Parzen probability density estimator is defined. Also, the limit power of the goodness-of-fit test constructed by means of this deviation is investigated. *© 2008 Bull. Georg. Natl. Acad. Sci.*

Key words: *distribution density, goodness-of-fit test, power, consistency, limit distribution.*

1. Limit distributions of some global measures of distributions of estimates $f_n(x)$ of the density $f(x)$ such as, for example, an integral square deviation constructed by means of the so-called weight function $W(x)$ not depending on *n* were studied in P. Bickel and M. Rosenblatt [1], E. Nadaraya ([2, 3]), P. Hall [4] and other works. .

The theory of the asymptotic behavior of an integral mean-square error

$$R(f_n, f; W_n) = E\int W_n(x)(f_n(x) - f(x))^2\, dx\,, \tag{1}$$

is developed in the work of T. T. Cai and M. G. Low [5], where $W_n(x) = a_n W(a_n(x-\ell_0))$, $\{a_n\}$ is a sequence of positive numbers, $W(x) \ge 0$ is a Borel-measurable function and ℓ_0 is some fixed point. If in (1) we put $W(x) = \frac{1}{2} I(-1 \le x \le 1)$ and pass to the limit as $a_n \to \infty$ for fixed *n*, then, roughly speaking, $R(f_n, f; W_n) \simeq E(f_n(\ell_0) - f(\ell_0))^2$. If, however, we put $a_n \equiv 1$ in (1) for all *n*, $\ell_0 = 0$ and assume that $W(x) \ge 0$ is an arbitrary bounded function, then $R(f_n, f; W_n) = E\|f_n - f\|^2_{L_2(W_n)}$. Thus the value $R(f_n, f; W_n)$ can be considered as *a generalization* of a measure of density estimation accuracy which contains a mean-square deviation of the estimate $f_n(x)$ of the density at the point and an integral mean-square deviation. Therefore it is natural to pose the question on the limit distribution of the value $\|f_n - f\|^2_{L_2(W_n)}$, $W_n(x) = a_n W(a_n(x-\ell_0))$. In the present paper this question is considered for the case where $f_n(x)$ is a nonparametric estimate of the Rosenblatt-Parzen density and $a_n \to \infty$ as $n \to \infty$. The case $a_n \to a_0 < \infty$ is of no interest because it follows from the results of the works [1-4].

Let $X_1, X_2, \ldots, X_n$ be independent, equally distributed random values having the unknown probability density function $f(x)$ and consider the Rosenblatt-Parzen nonparametric estimator $f_n(x)$ of the density $f(x)$

$$f_n(x) = \frac{\lambda_n}{n} \sum_{i=1}^{n} K(\lambda_n(x - X_i)),$$

where $K(x)$ is a function belonging to the class

$$H = \left\{ K : \int K(x) = 1,\ K(-x) = K(x),\ \sup_x |K(x)| < \infty,\ x^2 K(x) \in L_1(-\infty, \infty) \right\},$$

and $\{\lambda_n\}$ is a sequence of positive numbers converging to infinity.

Notation.

$$U_n = \frac{n}{\lambda_n} \|f_n - f\|^2_{L_2(W_n)}, \quad U_n^{(1)} = n\|f_n - Ef\|^2_{L_2(W_n)}, \quad \Delta_n = EU_n^{(1)},$$

$$\alpha_n(x, y) = \lambda_n [K(\lambda_n(x - y)) - EK(\lambda_n(x - X_1))],$$

$$\sigma_n^2 = 2\iint [E\alpha_n(u_1, X_1)\alpha_n(u_2, X_1)]^2 W_n(u_1) W_n(u_2) du_1 du_2,$$

$$\eta_{ij}^{(n)} = 2n^{-1}\sigma_n^{-1} \int \alpha_n(x, X_i)\alpha_n(x, X_j) W_n(x) dx,$$

$$\xi_j^{(n)} = \sum_{j=1}^{j-1} \eta_{ij}^{(n)}, \quad j = \overline{2, n}, \quad \xi_1^{(n)} = 0, \quad \xi_j^{(n)} = 0, \quad j > n,$$

$$\mathcal{F}_k^{(n)} = \sigma(\omega : X_1, X_2, \ldots, X_k).$$

Lemma 1. *The stochastic sequence* $\left(\xi_j^{(n)}, \mathcal{F}_j^{(n)}\right)_{j \geq 1}$ *is a difference-martingale.*

Lemma 2. *Let* $K(x) \in H$, $f(x) \in F$ *(F is the set of bounded functions on* $R = (-\infty, \infty)$ *which have bounded derivatives up to second order inclusive),* $W(x)$ *be bounded and* $W \in L_2(R)$. *If* $\lambda_n \to \infty$, $a_n \to \infty$ *and* $\frac{a_n}{\lambda_n} \to 0$ *as* $n \to \infty$, *then*

$$(\lambda_n a_n)^{-1} \sigma_n^2 \to 2f^2(\ell_0) \int K_0^2(z) dz \int W^2(v) dv, \quad K_0 = K * K, \quad f(\ell_0) \neq 0.$$

Theorem 1. *Let* $K(x) \in H$, $f(x) \in F$, $W(x)$ *be bounded and* $W \in L_1(R)$. *If* $\lambda_n \to \infty$, $a_n \to \infty$, $\frac{a_n}{\lambda_n} \to 0$ *and* $n^{-1}\lambda_n a_n^2 \to 0$ *as* $n \to \infty$, *then* $\sigma_n^{-1}\left(U_n^{(1)} - \Delta_n\right) \xrightarrow{d} N(0,1)$, *where d denotes the convergence in distribution, and* $N(0,1)$ *is a random value having a normal distribution with a zero mean value and variance* 1.

Proof. We have

$$\sigma_n^{-1}\left(U_n^{(1)} - \Delta_n\right) = \sqrt{\frac{n-1}{n}} H_n^{(1)} + H_n^{(2)}, \quad H_n^{(1)} = \sum_{j=1}^{n} \xi_j^{(n)},$$

and also

$$\operatorname{var} H_n^{(2)} = O\left(\frac{\lambda_n a_n}{n}\right) + O\left(n^{-1}\sigma_n^{-2}\right), \quad \text{i.e.} \quad H_n^{(2)} \xrightarrow{P} 0.$$

The asymptotic normality of $H_n^{(1)}$ takes place [6] if for each $\varepsilon \in (0,1]$ and $n \to \infty$

$$\sum_{k=1}^{n} E\left[\left(\xi_k^{(n)}\right)^2 I\left(\left|\xi_k^{(n)}\right| \geq \varepsilon\right) \Big/ \mathcal{F}_{k-1}^{(n)}\right] \xrightarrow{P} 0 \quad \text{(the Lindeberg condition)}$$

and

$$V_n^2 = \sum_{k=1}^{n} E\left(\left(\xi_k^{(n)}\right)^2 \Big/ \mathcal{F}_{k-1}^{(n)}\right) \xrightarrow{P} 1.$$

First let us verify that $V_n^2 \xrightarrow{P} 1$. For this, taking the definition of $\xi_j^{(n)}$ into account, we can write V_n^2 in the form

$$V_n^2 = \sum_{j=2}^{n} E\left(\sum_{i=1}^{j-1} \left(\eta_{ij}^{(n)}\right)^2 \Big| X_1, \ldots, X_{j-1}\right) + 2\sum_{j=2}^{n} E \sum_{i=1}^{j-1} \sum_{\ell=i+1}^{j-1} \left(\eta_{ij}^{(n)} \eta_{\ell j}^{(n)} \Big| X_1, \ldots, X_{j-1}\right) = V_{n1} + V_{n2}.$$

It is not difficult to show that

$$\operatorname{Var} V_{n1} = \frac{16\lambda_n^8}{n^4 \sigma_n^4} \left[\sum_{j=2}^{n} (j-1) E(\varepsilon_1 - \overline{\sigma}_n)^2 + 2\sum_{i=2}^{n} EZ_i^2 (n-i)\right] = B_{n1} + B_{n2},$$

and also

$$B_{n1} = O\left(\frac{a_n^2}{n^2}\right), \quad B_{n2} = O\left(\frac{a_n^2}{n}\right),$$

$$\varepsilon_i = \lambda_n^{-2} \iint \alpha_n(x, X_i)\alpha_n(y, X_i)\Phi_n(x, y)W_n(x)W_n(y)dxdy,$$

$$\Phi_n(x, y) = EK(\lambda_n(x - X_1))K(\lambda_n(y - X_1)) - EK(\lambda_n(x - X_1))EK(\lambda_n(y - X_1)),$$

$$Z_j = \sum_{i=1}^{j-1}(\varepsilon_i - \overline{\sigma}_n), \quad \overline{\sigma}_n = \iint \Phi_n^2(x, y)W_n(x)W_n(y)dxdy.$$

Therefore $\mathrm{Var}V_{n1} \to 0$. On the other hand, $EV_{n1} = 1 - \frac{1}{n} \to 1$. Therefore $V_{n1} \xrightarrow{P} 1$.

Now let us consider V_{n2}. Taking into account the inequality

$$E\left(\sum_{i=1}^{n} Y_j\right)^2 \le \left(\sum_{i=1}^{n}\left(EY_i^2\right)^{1/2}\right)^2$$

which is easy to verify and performing some simple calculations, we obtain

$$EV_{n2}^2 = O\left(\frac{a_n}{\lambda_n}\right).$$

Therefore $V_n^2 \xrightarrow{P} 1$.

Now we will establish the validity of the Lindeberg condition. For this, it suffices to make sure that $\sum_{j=1}^{n} E\left(\xi_j^{(n)}\right)^4 \to \infty$. Simple calculations show that

$$\sum_{j=1}^{n} E\left(\xi_j^{(n)}\right)^4 = O\left(\frac{a_n^2 \lambda_n}{n}\right).$$

Therefore

$$\sigma_n^{-1}\left(U_n^{(1)} - \Delta_n\right) \xrightarrow{d} N(0,1).$$

Theorem 2. *Let* $K(x) \in H$, $f(x) \in F$, $W(x)$ *be bounded,* $W(-x) = W(x)$, $x \in R$, *and* $x^2 W(x) \in L_1(R)$. *If* $\lambda_n \to \infty$, $a_n \to \infty$, $\frac{a_n}{\lambda_n} \to 0$, $\frac{\lambda_n a_n^2}{n} \to 0$ *and* $\lambda_n a_n^{-5} \to 0$, *then*

$$\left(\lambda_n a_n^{-1}\right)^{1/2} \sigma^{-1}(f)\left(U_n^{(2)} - \Delta(f)\right) \xrightarrow{d} N(0,1), \quad U_n^{(2)} = \lambda_n^{-1} U_n^{(1)},$$

$$\Delta(f) = f(\ell_0)\int K^2(u)du \int W(x)dx, \quad \sigma^2(f) = 2f^2(\ell_0)\int K_0^2(z)dz \int W^2(v)dv, \quad f(\ell_0) \ne 0.$$

Proof. Lemma 2, Theorem 1 and the representation $\Delta_n(f) = \lambda_n\left[\Delta(f) + O\left(a_n^{-2}\right) + O\left(\lambda_n^{-1}\right)\right]$ provide the proof of the theorem.

Theorem 3. *Let* $K(x)$, $f(x)$, $W(x)$ *satisfy the conditions of Theorem 2. If* $\lambda_n \to \infty$, $a_n \to \infty$, $\frac{a_n}{\lambda_n} \to 0$, $\frac{\lambda_n a_n^2}{n} \to 0$, $\lambda_n a_n^{-5} \to 0$, $\frac{\sqrt{n a_n}}{\lambda_n^{5/2}} \to 0$ *and* $n a_n^{-1/2} \lambda_n^{-9/2} \to 0$, *then*

$$\left(\lambda_n a_n^{-1}\right)^{1/2} \sigma^{-1}(f)\left(U_n - \Delta(f)\right) \xrightarrow{d} N(0,1).$$

Proof. We have

$$\left(\lambda_n a_n^{-1}\right)^{1/2}\left(U_n - U_n^{(2)}\right) = \sqrt{\frac{\lambda_n}{a_n}}\left(\Theta_n + R_n\right),$$

$$\Theta_n = \frac{n}{\lambda_n}\int \left(Ef_n(x) - f(x)\right)^2 W_n(x)dx,$$

$$R_n = 2\frac{n}{\lambda_n}\int \left(f_n(x) - Ef_n(x)\right)\left(Ef_n(x) - f(x)\right)W_n(x)dx.$$

By virtue of the generalized Minkovski inequality and

$$\max_x \left|Ef_n(x)-f(x)\right| = O\left(\lambda_n^{-2}\right),$$

we obtain

$$\left(\lambda_n a_n^{-1}\right)^{1/2} E\left|R_n\right| = O\left(\sqrt{na_n}\,\lambda_n^{-5/2}\right)$$

and also

$$\left(\lambda_n a_n^{-1}\right)^{1/2} \Theta_n = O\left(na_n^{-1/2}\lambda_n^{-9/2}\right).$$

The theorem is proved.

2. The assertion of Theorem 3 enables us to construct goodness-of-fit tests of the asymptotic level α for testing the hypothesis $H_0 : f(x)=f_0(x),\ f_0(\ell_0)\neq 0$. For this it is necessary to reject H_0 if

$$U_n \geq d_n(\alpha)=\Delta(f_0)+\left(\frac{\lambda_n}{a_n}\right)^{-1/2}\cdot\varepsilon_\alpha\sigma(f_0), \tag{*}$$

where ε_α is the quantile of the level α of a standard normal distribution.

Theorem 4. *Let all the conditions of Theorem* 3 *be fulfilled. Then* $\Pi_n(f_1)=P_{H_1}\{U_n \geq d_n(\alpha)\}\to 1$ *as* $n\to\infty$. *Therefore the goodness-of-fit defined in* (*) *is consistent against any alternative* $H_1 : f(x)=f_1(x),\ f_1(x)\neq f_0(x)$ *on the set of a positive Lebesgue measure,* $f_1(\ell_0)\neq f_0(\ell_0)$.

It is not difficult to show that

$$\Pi_n(f_1)=P_{H_1}\left\{\left(\lambda_n a_n^{-1}\right)^{1/2}\sigma^{-1}(f_1)\left(U_n^*-\Delta(f_1)\right)\geq -\frac{n}{\sqrt{\lambda\, a_n}}\left(\sigma^{-1}(f_1)R_n+o_p(1)\right)\right\},$$

$$U_n^* = n\lambda_n^{-1}\left\|f_n-f_1\right\|^2_{L_2(W_n)}.$$

Since for the hypothesis H_1 we have

$$\sqrt{\frac{\lambda_n}{a_n}}\sigma^{-1}(f_1)\left(U_n^*-\Delta(f_1)\right)\xrightarrow{d} N(0,1),\quad n\lambda_n^{-1/2}a_n^{-1/2}\to\infty$$

and

$$R_n\to\left(f_1(\ell_0)-f_0(\ell_0)\right)^2\int W(x)dx>0$$

we conclude that $\Pi_n(f_1)\to 1$.

Now let us introduce into the consideration the sequences of locally close alternatives ([7], [8])

$$H_{1n} : f_{1n}(x)=f_0(x)+\alpha_n\varphi\left(\frac{x-\ell_n}{\gamma_n}\right)+o(\alpha_n\gamma_n),$$

$$\ell_n=\ell_0+o(\gamma_n),\quad \varphi(x)\in F,\quad \int\varphi(x)dx=0.$$

Theorem 5. *Let* $K(x)$, $f_{1n}(x)$, $W(x)$, λ_n *and* a_n *satisfy the conditions of Theorem* 3. *Let, in addition,* $W(x)$ *be continuous at the point* 0 *and* $W(0)>0$, $\alpha_n\gamma_n=o\left(n^{-1/2}\right)$, $n\lambda_n^{-1/2}a_n^{1/2}\gamma_n\alpha_n^2\to\gamma_0>0$, $\lambda_n a_n^{-1}\alpha_n^2\to 0$, $\lambda_n\gamma_n\to\infty$, $\alpha_n^{-1}\lambda_n^{-2}\to 0$ *and* $a_n\gamma_n\to 0$ *as* $n\to\infty$. *Then*

$$P_{H_{1n}}\{U_n\geq d_n(\alpha)\}\to 1-\Phi\left(\varepsilon_\alpha-\gamma_0 W(0)\sigma^{-1}(f_0)\int\varphi^2(x)dx\right).$$

Proof. We have

$$P_{H_{1n}}\{U_n\geq d_n(\alpha)\}=$$

$$=P_{H_{1n}}\left\{\sqrt{\frac{\lambda_n}{a_n}}\left(U_n^{(3)}-\Delta(f_{1n})\right)\sigma^{-1}(f_{1n})\geq\frac{\sigma(f_0)}{\sigma(f_{1n})}\varepsilon_\alpha+\sqrt{\frac{\lambda_n}{a_n}}\sigma^{-1}(f_{1n})\left[\Delta(f_0)-\Delta(f_{1n})-A_{1n}+A_{2n}\right]\right\},$$

$$U_n^{(3)}=n\lambda_n^{-1}\left\|f_n-f_{1n}\right\|^2_{L_2(W_n)},$$

$$A_{1n}=n\lambda_n^{-1}\left\|f_{1n}-f_0\right\|^2_{L_2(W_n)},\qquad A_{2n}=n\lambda_n^{-1}\int\left(f_n(x)-f_{1n}(x)\right)\left(f_{1n}(x)-f_0(x)\right)W_n(x)dx.$$

From Theorem 3 it follows that

$$\left(\lambda_n a_n^{-1}\right)^{1/2}\left(U_n^{(3)}-\Delta\left(f_{1n}\right)\right)\sigma^{-1}\left(f_{1n}\right)\xrightarrow{d} N(0,1)$$

for the hypothesis H_{1n}. Let us now show that

$$\sqrt{\frac{\lambda_n}{a_n}}\sigma^{-1}\left(f_{1n}\right)A_{2n}\xrightarrow{P} 0 .$$

Indeed,

$$\sqrt{\frac{\lambda_n}{a_n}}E\left|A_{2n}\right|\le L_n^{(1)}+L_n^{(2)};$$

also

$$L_n^{(2)}=O\left(\alpha_n^{-1}\lambda_n^{-2}\right)$$

and

$$L_n^{(1)}\le cna_n^{1/2}\lambda_n^{-1/2}\alpha_n\times$$

$$\times\left\{\frac{1}{n}\int f(u)\varphi^2\left(\frac{u-\ell_n}{\gamma_n}\right)du+\gamma_n^{-2}n^{-1}\lambda_n^{-2}\int f(u)du\left[\int\int_0^1|t||K(t)|\left|\varphi^{(1)}\left(\frac{u-\ell_n}{\gamma_n}\right)+\frac{zt}{\lambda_n\gamma_n}\right|dt\,dz\right]^2\right\}^{1/2}.$$

Hence by virtue of the generalized Minkovski inequality we obtain

$$L_n^{(1)}=O\left(\lambda_n^{-1/4}a_n^{1/4}\right)+O\left(\gamma_n^{-1}\lambda_n^{-1}\left(\frac{a_n}{\lambda_n}\right)^{1/4}\right).$$

Therefore

$$\sqrt{\frac{\lambda_n}{a_n}}E\left|A_{2n}\right|=O\left(\left(\frac{a_n}{\lambda_n}\right)^{1/4}\right)+O\left(\alpha_n^{-1}\lambda_n^{-2}\right).$$

Furthermore, using the condition

$$n\lambda_n^{-1/2}a_n^{1/2}\gamma_n\alpha_n^2\to\gamma_0>0$$

it is not difficult to establish that

$$\sigma^{-1}\left(f_{1n}\right)\sqrt{\frac{\lambda_n}{a_n}}A_{1n}\to\gamma_0 W(0)\sigma^{-1}\left(f_0\right)\int\varphi^2(u)\,du,\quad W(0)\ne 0 .$$

The theorem is proved.

The conditions of the theorem as regards λ_n, a_n, α_n and γ_n are fulfilled if, for example, we assume that $\lambda_n=n^{\delta}$, $a_n=n^{\varepsilon}$, $\alpha_n=n^{-\alpha}$, $\gamma_n=n^{-\beta}$ for $\alpha=\frac{9}{35}$, $\beta=\frac{2}{7}$, $\delta=\frac{2}{5}+\varepsilon$, $\frac{1}{10}<\varepsilon<\frac{1}{5}$; $\alpha=\frac{11}{30}$, $\beta=\frac{1}{6}$, $\delta=\frac{1}{5}+\varepsilon$, $\frac{1}{20}<\varepsilon<\frac{1}{6}$ and so on.

It is well-known that for some α, β and δ, for which $\alpha+\beta>1/2$, $1-2\alpha-\beta=\delta/2$, the limit power of the Rosenblatt-Bickel goodness-of-fit ([2], [7], [8])

$$T_n\ge\int f_0(x)w(x)\,dx\int K^2(u)\,du+\lambda_n^{-1/2}\varepsilon_\alpha\sigma_0 ,$$

$$T_n=n\lambda_n^{-1}\int\left(f_n(x)-f_0(x)\right)^2w(x)\,dx, \qquad (**)$$

$$\sigma_0^2=2\int f_0^2(x)w^2(x)\,dx\int K_0^2(x)\,dx$$

used for testing the hypothesis $H_0: f(x)=f_0(x)$ against the alternative

$$H_{1n}: f_{1n}(x)=f_0(x)+\alpha_n\varphi\left(\frac{x-\ell_n}{\gamma_n}\right),\quad \ell_n=\ell_0+o(\gamma_n)$$

$$\left(\lambda_n=n^{\delta},\ \alpha_n=n^{-\alpha}\ \text{ and }\ \gamma_n=n^{-\beta}\right)$$

is equal to

$$\gamma(T)=1-\Phi\left(\varepsilon_\alpha-\frac{w(\ell_0)}{\sigma_0}\int\varphi^2(u)\,du\right),$$

while the limit power $\gamma(U)$ of the goodness-of-fit (*) is equal to one for $a_n=n^{\varepsilon}$, $0<\varepsilon<\delta$. Further, for some α, β, δ and ε, for which $\alpha+\beta>1/2$, $1-2\alpha-\beta+\varepsilon/2=\delta/2$, the limit power of the goodness-of-fit (*) is equal by virtue of Theorem 5 to

$$\gamma(u)=1-\Phi\left(\varepsilon_\alpha-\frac{w(0)}{\sigma(f_0)}\int\varphi^2(u)\,du\right),$$

while the limit power $\gamma(T)$ of the goodness-of-fit (**) is equal to $1-\Phi(\varepsilon_\alpha)$. Moreover, the calculation of the right-hand side of (*) becomes essentially simpler as compared with (**) and therefore when choosing between the goodness-of-fit tests we will give preference to the goodness-of-fit test based on U_n.

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ე. ნადარაია

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