ABSTRACT. In the beginning (of the author’s scientific career) there was (Prof. Nikoloz Ivanesdze) Muskhelishvili. His method, based on the theory of complex variables, enabled the author to solve the problem of a simply supported trapezium-shaped plate with right angles subjected to a uniformly distributed static load. An essential feature of the solution of this problem, in the framework of a doctoral dissertation, was conformal mapping of the unit circle onto a trapezium with right angles representing a quarter of a regular hexagon. The purpose of the brief description of this work, written by a scientific novice approximately forty years ago, is to pay late tribute to the great Georgian scientist Muskhelishvili.

Key words: Nikoloz Muskhelishvili.

1. Introduction

At the end of October 1969, I submitted my doctoral dissertation “On the Analysis of Trapezium-shaped Plates with Right Angles by Means of Muskhelishvili’s Method” [1] to Vienna University of Technology (TUW) (Fig. 1).

This topic was proposed by the Head of the Institute for Strength of Materials at the Department of Civil Engineering of TUW, Prof. Walter Mudrak. It was motivated by consulting work consisting of the analysis of deformations and stresses of an elastic plate of the aforementioned shape, subjected to a uniformly distributed static load. Such plates can be found, e.g., as load-carrying components of structures at skew intersections of two roads (Fig. 2).

The method of solution of the problem was left open. A senior colleague suggested Muskhelishvili’s method to me. With the help of conformal mapping of the unit circle onto the trapezium this method would allow reducing the solution to the one for a circular plate.

At that time, approximately forty years ago, I knew little about the famous Georgian scientist Prof. Nikoloz Ivanesdze Muskhelishvili (Fig. 3), let alone his opus magnum “Some Basic Problems of the Mathematical Theory of Elasticity” [2]. Soon, however, this fundamental work became scientific gospel to me. The recollection of this time prompted me to modify the initial sentence in the prologue of St. John’s gospel to “In principio erat Muskhelishvili” for use as the title of this article. Indeed, in the beginning of my scientific career there was Muskhelishvili whose “verba mathematica” I was studying with great enthusiasm.

I consider the honorable invitation by the President of the Georgian Academy of Sciences, Prof. Tamaz Gamkrelidze, to write a paper on my early experience with Muskhelishvili’s method as a lucky coincidence allowing me to pay late tribute to the great Georgian Academician Muskhelishvili by means of my earliest scientific work.
As a doctoral student my interest in biographical details of scientists was relatively small. All I knew about Prof. Muskhelishvili was that he published the aforementioned book and also wrote a monography on “Singular Integral Equations” [3]. I did not know e.g. that he was a student of physics and mathematics and a postgraduate, respectively, of St. Petersburg University, where, in 1934, he obtained the degree of Dr. of Physical and Mathematical Sciences. With increasing age, my interest in N. I. Muskhelishvili’s biography has grown. What I have always admired, is his tenure of office as Professor at Tbilisi State University from 1922 until his death in 1976, as Director of the Mathematical Institute of the Academy of Sciences of the Georgian SSR from 1941 until his death, and, last but not least, as President of the Academy from 1941-1972.

To indulge in further details of the biography of the great Georgian scientist Muskhelishvili in the Bulletin of the Georgian Academy of Sciences would be carrying coals to Newcastle. Georgian readers, however, may rest assured that Prof. Muskhelishvili’s membership in several foreign scientific academies of high reputation, the host of important prizes and awards bestowed on him, and his extensive public activities are just as impressive from my point of view as former Secretary General and President of the Austrian Academy of Sciences who had to do with many biographies of researchers, as from the vantage point of compatriotic scientists.

At the time of my doctoral studies at TUW, Muskhelishvili’s method was the only promising analytical tool for determination of displacements and stresses of a trapezium-shaped plate with right angles. Twenty five years later, in a handbook chapter on plates and shells [4] serving as lecture notes at TUW, Muskhelishvili’s method was employed for the solution of the most difficult plate problems in a hierarchy of increasing level of such problems.

When I was writing my Viennese doctoral dissertation, the Finite Element Method (FEM) was yet not taught at Austrian universities. Soon after graduation to Dr.techn. I realized the necessity to study this powerful computer-oriented method which revo-
lutionized, among other things, structural analysis. Thirty three years after completion of Ph.D. studies in the USA with a dissertation on the analysis of doubly corrugated shells by the FEM [5], I daresay that the impressive versatility and great efficiency of this numerical method which has dominated my scientific career is largely compensated by the mathematical beauty of Muskhelishvili’s analytical method.

3. Analysis of elastic plates by means of Muskhelishvili’s method

3.1. Formulation of the boundary value problem. The general solution of the inhomogeneous bi-potential equation, describing the deflections of an elastic plate, can be written as [2]

\[ w = \text{Re}[\bar{\varphi}(z) + \chi(z)] + w_p, \]  

(1) 

where \( \text{Re}[\cdot] \) is the real part of the complex function in brackets, \( \varphi \) and \( \chi \) denote two analytic functions of the complex variable \( z = x + iy \) with \( i = \sqrt{-1} \), \( \bar{z} \) stands for the conjugate complex variable, and \( w_p \) symbolizes a particular solution of the inhomogeneous bi-potential equation. For 

\[ \varphi(0) = 0 \quad \text{and} \quad \text{Im} \chi(0) = 0, \]  

(2) 

where \( \text{Im} \chi(0) \) is the imaginary part of \( \varphi(0) \), a rigid body motion of the plate is impossible, resulting in a unique solution [1].

For the special case of a uniformly distributed load \( p \), \( w_p \) is given as [2]

\[ w_p = \frac{p(z \bar{z})}{64 K} \quad \text{with} \quad K = \frac{Eh^3}{12(1-\nu^2)}, \]  

(3) 

as the stiffness of the plate, where Young’s modulus \( E \), Poisson’s ratio \( \nu \), and the thickness \( h \) are constant quantities.

For the special case of simply supported plates with straight boundaries, the Navier boundary conditions hold:

\[ w = 0, \quad \Delta w = 0 \]  

(4) 

with \( \Delta \) denoting the Laplace operator.

Substituting \( z = z_R \), where the subscript \( R \) stands for “boundary”, into (1) and (3) and inserting the obtained relations into (4), yields 

\[ \frac{p(z_R \bar{z}_R)}{64 K} = -\frac{1}{2} \left[ \bar{z}_R \varphi(z_R) + z_R \bar{\varphi}(z_R) + \chi(z_R) + \bar{\chi}(z_R) \right], \]  

(5a) 

\[ \frac{p(z_R \bar{z}_R)}{4 K} = -2 \left[ \varphi'(z_R) + \bar{\varphi}'(z_R) \right]. \]  

(5b) 

Note that (5b) does not depend on \( \chi(z_R) \).

3.2. Solution of the boundary value problem for circular plates. Expansion of the analytic functions \( \varphi(z) \) and \( \chi(z) \) as power series yields [2]

\[ \varphi(z) = \sum_{k=1}^{\infty} a_k z^k \quad \text{and} \quad \chi(z) = \sum_{k=0}^{\infty} a_k z^k. \]  

(6)
Expansion of the right-hand side of (3) as a Fourier series for $z = z_R$ gives

$$w_p(z_R) = \sum_{k=-\infty}^{\infty} A_k e^{ik\vartheta},$$  

(7)

where $A_k$ are the known Fourier coefficients and $\vartheta$ is the polar angle. With

$$z_R = Re^{i\vartheta} = R\sigma \quad \text{and} \quad \bar{z}_R = Re^{-i\vartheta} = R\sigma^{-1}$$  

(8)

the boundary conditions (5a) and (5b) become

$$\sum_{k=-\infty}^{\infty} A_k \sigma^k = -\frac{1}{2} \left( R\sigma^{-1} \sum_{k=1}^{\infty} a_k R_k \sigma^{-k} + R\sigma \sum_{k=1}^{\infty} \bar{a}_k R_k \sigma^{-k} + \sum_{k=0}^{\infty} a'_k R_k \sigma^k + \sum_{k=0}^{\infty} \bar{a}'_k R_k \sigma^{-k} \right),$$  

(9a)

$$\sum_{k=-\infty}^{\infty} B_k \sigma^k = -2 \left( \sum_{k=1}^{\infty} a_k k R_{k-1} \sigma^{-k-1} + \sum_{k=1}^{\infty} \bar{a}_k k R_{k-1} \sigma^{-(k-1)} \right),$$  

(9b)

where the left-hand side of (9b) represents a Fourier series expansion of $\Delta w_p(z_R)$ with $B_k$ as known Fourier coefficients.

The unknown coefficients $a_k$, $\bar{a}_k$, $a'_k$ and $\bar{a}'_k$ are obtained by means of comparison of coefficients.

Because of the curved boundary, the boundary condition (4,2) yields

$$m_n(z_R) = \frac{K}{R} (1-\nu) \frac{\partial w}{\partial R} \neq 0,$$  

(10)

where $m_n(z_R)$ is the bending moment normal to the boundary.

3.3. Solution of the boundary value problem by means of conformal mapping of the unit circle onto the desired domain. With the help of the function

$$z = \omega(\zeta),$$  

(11)

mapping the unit circle $|\zeta| < 1$ conformally onto the interior of the domain representing the desired shape of the plate, the boundary value problem can be solved analogous to the one for circular plates. Expressing $\varphi(z)$ and $\chi(z)$ by means of (11) as

$$\varphi(z) = \varphi(\omega(\zeta)) = \varphi_1(\zeta) \quad \text{and} \quad \chi(z) = \chi(\omega(\zeta)) = \chi_1(\zeta)$$  

(12)

and specializing (8) for the boundary of the unit circle ($R = 1$, $z_R = \zeta_R$) to obtain $\zeta_R = \sigma$, the boundary conditions (5a) and (5b) can be rewritten as [2]

$$p \frac{\omega(\sigma)\overline{\omega(\sigma)}}{64K} = -\frac{1}{2} \left[ \omega(\sigma)\varphi_1(\sigma) + \omega(\sigma)\overline{\varphi_1(\sigma)} + \chi_1(\sigma) + \overline{\chi_1(\sigma)} \right],$$  

(13a)

$$p \frac{\omega(\sigma)\overline{\omega(\sigma)}}{4K} = -2 \left[ \frac{\varphi_1(\sigma)}{\omega'(\sigma)} + \frac{\overline{\varphi_1(\sigma)}}{\overline{\omega'(\sigma)}} \right],$$  

(13b)

where

$$\frac{d\varphi_1}{d\zeta} = \frac{d\varphi}{dz} \frac{dz}{d\zeta} = \frac{d\varphi}{dz} \frac{d\omega}{dz} \frac{dz}{d\zeta}.$$  

(14)

Since the mapping function is a regular function, it can be expanded, within its convergence circle, as a power series:

$$\omega(\zeta) = \sum_{k=0}^{\infty} b_k \zeta^k.$$  

(15)
Assuming that this series converges for \( \zeta_m = \sigma \),

\[
\omega(\sigma) = \sum_{k=0}^{\infty} b_k \sigma^k.
\]  

(16)

The solution of the boundary value problem is similar to the one for circular plates. Details of this solution are given in [2]. Its result are the coefficients \( a_n, b_n, a_k \), and \( \bar{a}_k \) of the power series

\[
\varphi_1(\zeta) = \sum_{k=1}^{\infty} a_k \zeta^k, \quad \varphi_1(\zeta) = \sum_{k=1}^{\infty} a_k \zeta^k, \quad \chi_1(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k, \quad \chi_1(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k,
\]

(17)

which differ from the ones in (9b).

4. Example. Trapezium-shaped plate with right angles

4.1. Choice of special geometric shape. The geometric shape of the plate is chosen as a quarter of a regular hexagon (Fig. 4). Hence, the proportion of the side lengths is \( a : b : c : d = 1:1:1/2: \sqrt{3}/2 \).

The reason for this choice is that the trapezium-shaped plate concerned may be viewed as the “most characteristic” representative of this class of plates. “Less characteristic” representatives are shown in Fig. 5.

For \( c/a > 1/4 \), Bareš suggested to replace the trapezium-shaped plate by a rectangular plate with the side lengths

\[
a_1 = \frac{2}{3} \frac{2c + a}{a + c}, \quad b_1 = d = \frac{a(a - c)}{6(a + c)}.
\]

(18)

If, by comparison of results for \( c/a = 1/2 \), this suggestion should prove to be useful, the more this would be the case for \( c/a > 1/2 \).

4.2. Conformal mapping of the unit circle onto the chosen trapezium with right angles. At first, the unit circle is mapped conformally onto a quarter circle with the same radius (Fig. 6). The mapping function is obtained as [2, 7]

\[
\rho = \frac{1 + i\zeta - \sqrt{2} \sqrt{1 - \zeta^2}}{1 + \zeta}.
\]

(19)

With the help of the reflection principle by Schwarz [7], the quarter circle is then mapped conformally onto the chosen trapezium (Fig. 7). According to this principle, the mapping function is the same as the one for conformal mapping of the unit circle onto a rectangular hexagon, obtained as [2, 8]

\[
\frac{dz}{d\rho} = A(1 - \rho^6)^{1/3} \quad \text{with} \quad A = 0.8986.
\]

(20)

The right-hand sides of (19) and (20) are expanded as power series. Substitution of the power series for \( \rho(\zeta) \) into the one for \( z(\rho) \), obtained by termwise integration of the power series for \( dz/d\rho \), yields
\[ z = \omega(\zeta) = A \sum_{k=0}^{\infty} \gamma_k \zeta^k. \]  

(21)

Figure 6. Conformal mapping of the unit circle onto a quarter circle with the same radius.

Figure 7. Conformal mapping of the quarter circle onto the chosen trapezium with the help of the reflection principle by Schwarz.

Figure 8 illustrates corresponding points of the unit circle and of an approximation of its conformal mapping onto the chosen trapezium obtained by truncation of the power series in (21) after the term \( \gamma_1 \zeta \). In [2], this approximation was improved by an estimation of the residue. From the viewpoint of engineering design, however, such an improvement is not necessary.

Because of the truncation of (21), the actual boundary of the conformal mapping of the unit circle, as opposed to the desired boundary, is curved. Hence, satisfaction of the boundary condition \( \Delta w = 0 \) (4) results in violation of the

Figure 8. Corresponding points of the unit circle and its conformal mapping onto an approximation of the chosen trapezium.

boundary condition $m_n = 0$, analogous to $m_n (z) = 0$ (10) for a circular plate. If, instead of $\Delta w = 0$, $m_n = 0$ had been chosen as boundary condition, its satisfaction for the actual boundary would not necessarily have been better than satisfaction of $\Delta w = 0$ for this boundary.

4.3. Selected distributions of displacements for the chosen trapezium-shaped plate. Fig. 9 shows plots of the dimensionless term $(8K/p \ell^4_{DA}) w$, where the so far undefined quantity $\ell_{DA}$ denotes the length of the longer one of the two parallel sides of the trapezium-shaped plate. The mapping of a specific diameter of the unit circle is defined by the angle $\vartheta$ (Fig. 8). Because of the truncation of the infinite series (21), the range of validity of the curves in Fig. 9 is restricted to their center parts shown as full lines.

For $v = 0.3$, the results for the decisive bending moments $m_x$ and $m_y$ obtained by means of approximate analysis of the trapezium-shaped plate as a rectangular plate with side lengths $a$ and $b$, according to (18) are on the safe side [2]. Nevertheless, since the difference is only 8.9% and 12.3%, respectively, the approximate analysis is not uneconomical.

5. Conclusions

In principio erat Muskhelishvili. In the beginning of my scientific career, in the late sixties of the past century, there was the scientific icon Prof. Nikoloz Ivaniedze Muskhelishvili. The mathematical and mechanical brilliance of his great works has fascinated me ever since. My intensive preoccupation with Computational Mechanics, in general, and with the FEM, in particular, has had no influence on my reverence for the great Georgian mathematical analyst Muskhelishvili, to whom I wanted to pay late tribute by this paper.

Looking back at forty years of scientific work in the area of Solid Mechanics, a remark by John F. Kennedy at the award of an honorary doctorate from Yale University came to my mind: “I have the best of both worlds, a Harvard education and a Yale degree.” Let me conclude these autobiographical reflections by claiming that also I, though in a somewhat different manner, have the best of two worlds, of the Old and the New World: an Austrian Dr.techn. dissertation based on fundamental work by the Georgian scientist Muskhelishvili and an American Ph.D. thesis wherein the FEM was used.

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«In principio erat Muskhelishvili»

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