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Mathematics

General Solutions of Linear Matrix Canonical Second Order Differential Equations with Variable Coefficients

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ABSTRACT. This article contains the formulae of general solutions for particular classes of nonhomogeneous ordinary linear second order matrix differential equations with variable coefficients. *© 2007 Bull. Georg. Natl. Acad. Sci.*

Key words: *matrix differential equations, variable coefficients, general solutions.*

1. Canonical Equation. Problem Statement

Let linear matrix canonical second order differential equation be called an equation

$$(Ep-A)[(Ep-B)X]=F, \quad p=\frac{d}{dt}, \tag{1}$$

where

$$A=A(t)=(a_j^i(t)),\ B=B(t)=(b_j^i(t)),\ pB,\ F=F(t)=(f_j^i(t)),\ \ i,j=\overline{1n},\ t\in I=[t_1,t_2]\subset]-\infty,+\infty[$$

are given matrices with continuous elements; E – unit matrix.

It is obvious that equation (1) is equivalent to the equation

$$\ddot{X}-(A+B)\dot{X}+(AB-\dot{B})X=F, \quad t\in I. \tag{2}$$

Here and hereinafter the dot over functions means differentiation with respect to t by corresponding number.

Further everywhere an admissible function will be called any function $u=u(t)$, $t\in I$, in respect to which operations presented in the article are valid on the whole interval I.

Definition 1. The solution of equation (1) will be called matrix function $X=X(t)$ defined on interval I, substitution of which in the equation (1) is admissible as a result of which we get the identity.

$$(Ep-A)[(Ep-B)X(t)]\equiv F, \quad t\in I.$$

Definition 2. Let t_o be arbitrary fixed point of the interval I and X_0, X_1 be arbitrary fixed constants of matrix $n\times n$. Matrix function $X(t, C_0, C_1)$ defined on the interval I and depending on arbitrary constants C_0, C_1 of matrix $n\times n$ will be called the general solution of equation (1), if $X(t, C_0, C_1)$, $t\in I$ is a solution of equation (1) satisfying the initial conditions

$$X(t_0, C_0, C_1)=X_0,\ \ \dot{X}(t_0, C_0, C_1)=X_1. \tag{3}$$

The basic problem consists in constructing the general equation of type (1).

2. Regular Matrices. The Main Theorem

To construct the general solution of equation (1) we shall need matrix function of regular matrix.

Definition 3. Matrix G=$G(t)$=$(g_j^i(t))$ with continuous elements $g_j^i(t),\ i, j = \overline{1n},\ t \in I,$ will be called a regular matrix if there exists a matrix function $e^{\int Gdt}$ definite, continuous and continuously differentiable with respect to t on the interval I, satisfying the conditions:

$$\frac{d}{dt} e^{\int Gdt} = Ge^{\int Gdt},\ t \in I, \tag{4}$$

$$\exists e^{-\int Gdt}, e^{-\int Gdt} e^{\int Gdt} = e^{\int Gdt} e^{-\int Gdt} = \mathrm{E},\quad t \in I. \tag{5}$$

Remark. Generally speaking, from regularity of matrix G the regularity of matrix -G does not follow. Really

$$0 = \frac{d}{dt}(e^{\int Gdt} e^{-\int Gdt}) = Ge^{\int Gdt} e^{-\int Gdt} + e^{\int Gdt} \frac{d}{dt} e^{-\int Gdt} = G + e^{\int Gdt} \frac{d}{dt} e^{-\int Gdt}.$$

Consequently

$$\frac{d}{dt} e^{-\int Gdt} = -e^{-\int Gdt} G,$$

i.e. if matrices $e^{-\int Gdt}$ and G are non-permutable, the formula (4) is not fulfilled.

The following is valid.

Theorem 1 (Basic Theorem). *If matrices A=$A(t)$, B=$B(t)$, $t\in I$ are regular and matrix B is continuously differentiable on the interval I, then general solution of equation* (2) *or that of equivalent equation* (1) *has the form*

$$X = e^{\int Bdt}(C_0 + \int e^{-\int Bdt} \mathrm{Y} dt),$$

where

$$\mathrm{Y} = e^{\int Adt}(C_1 + \int e^{-\int Adt} Fdt),\ \ t \in I. \tag{6}$$

Here C_0 and C_1 are arbitrary constants of matrix n x n.

Validity of Theorem 1 is directly verified.

Theorem 2. *Let us consider the equation*

$$\ddot{X} + A(t)\dot{X} + B(t)X = F(t),\ \ t \in I, \tag{7}$$

where $A(t)$, $B(t)$, $F(t)$ are n x n-matrices, definite and continuous on the interval I. If there exists n x n regular matrix $\chi = \chi(t),\ t \in I,$ which satisfies the condition

$$\dot{\chi} + \chi^2 + A(t)\chi + B(t) \equiv 0,\quad t \in I, \tag{8}$$

and matrix Ω= – ($\chi(t)$+$A(t)$), $t\in I$ is regular, then general solution of equation (7) *has the form*

$$X = e^{\int \chi dt}(C_0 + \int e^{-\int \chi dt} \mathrm{Y} dt),$$

where

$$\mathrm{Y} = e^{\int \Omega dt}(C_1 + \int e^{-\int \Omega dt} Fdt),\ \ t \in I. \tag{9}$$

Here C_0 and C_1 are arbitrary constants of matrix n x n.

Remark. Equation $\dot{\chi}+\chi^2+A(t)\chi+B(t)=0,\ t\in I,$ is matrix analogue of Riccati equation.

Validity of Theorem 2 is directly verified.

From Theorem 2 it follows

Theorem 3. *Let us consider the equation*

$$\ddot{X}+B(t)X=F(t),\ t\in I, \tag{10}$$

where $B(t)$, $F(t)$ are n x n-matrices, definite and continuous on the interval I. If there exists n x n regular matrix $\chi=\chi(t)$, $t\in I$, satisfying the condition

$$\dot{\chi}+\chi^2+B(t)\equiv 0,\ t\in I \tag{11}$$

and matrix $-\chi=-\chi(t),\ t\in I$ is regular, then general solution of equation (10) has the form

$$X=e^{\int\chi dt}(C_0+\int e^{-\int\chi dt}\mathrm{Y}dt),$$

where

$$\mathrm{Y}=e^{-\int\chi dt}(C_1+\int e^{\int\chi dt}Fdt),\quad t\in I. \tag{12}$$

Here C_0 and C_1 are arbitrary constants of matrix n x n.

Consider the case when $B(t)=-C^2$, where C is an arbitrary constant of n x n matrix. Then a constant matrix $\chi=C$ satisfies the condition (11). Consequently, there exist matrix functions $e^{t\chi}$ and $e^{-t\chi}$ satisfying the conditions

$$\frac{d}{dt}e^{t\chi}=\chi e^{t\chi},\ \frac{d}{dt}e^{-t\chi}=-\chi e^{-t\chi},\ t\in I\ \text{(see [1])}.$$

Thus, the conditions of Theorem (3) are fulfilled and general solution of equation

$$\ddot{X}-C^2X=F(t),\ t\in I, \tag{13}$$

has the form

$$X=e^{tC}(C_0+\int e^{-tC}\mathrm{Y}dt),$$

where

$$\mathrm{Y}=e^{-tC}(C_1+\int e^{tC}Fdt),\ t\in I. \tag{14}$$

In the special case when $C=iE$, equation (13) gets the form

$$\ddot{X}+X=F(t),\quad t\in I,$$

and general solution of this equation by virtue of formula (14) has a form

$$X=e^{it}(C_0+\int e^{-it}\mathrm{Y}dt),$$

where

$$\mathrm{Y}=e^{-it}(C_1+\int e^{it}Fdt),\ t\in I.$$

3. Spectral Matrices. Regularity Condition

Consider matrix $A=A(t)=(a^i_j(t))$, the elements of which $a^i_j = a^i_j(t),\ i,j=\overline{1n},$ are arbitrary admissible functions defined on the interval I.

Definition 4. Characteristic polynomial of matrix $A=A(t)$, $t\in I$ will be called polynomial of the n degree with respect to λ with variable coefficients depending on t and defined by the formula

$$P(\lambda, A)=\det(\lambda E-A(t)),\quad t\in I.$$

Below we give definition of the spectral matrix playing the leading role.

Definition 5. Matrix $A=A(t)$, $t\in I$, will be called spectral matrix if the following conditions are fulfilled.

$$P(\lambda, A)=(\lambda-\lambda_1(t))^{k_1}...(\lambda-\lambda_m(t))^{k_m},\quad t\in I,$$

where $\lambda_i(t)\neq\lambda_j(t),\ i\neq j,\ t\in I,\ k_1,...,k_m$ are natural numbers, $k_1+...+k_m=n$.

Determinant

$$\Lambda(A)=\begin{vmatrix} \lambda_1^{n-1}(t),\ \lambda_1^{n-2}(t),\ \ldots\ldots\ldots\ldots, & \lambda_1(t), & 1 \\ (n-1)\lambda_1^{n-2}(t),\ \ (n-1)(n-2)\lambda_1^{n-3}(t),\ \ldots\ldots, & 2\lambda_1(t),\ \ 1, & 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & \ldots \\ (n-1)...(n-k_1+1)\lambda_1^{n-k_1}(t),\ \ ...,k_1!\lambda_1(t),(k_1-1)!, & 0,\ldots\ldots, & 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & \ldots \\ \lambda_m^{n-1}(t),\ \lambda_m^{n-2}(t),\ \ldots\ldots\ldots\ldots, & \lambda_m(t), & 1 \\ (n-1)\lambda_m^{n-2}(t),(n-1)(n-2)\lambda_m^{n-3}(t),\ \ldots\ldots, & 2\lambda_m(t),\ \ 1, & 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & \ldots \\ (n-1)....(n-k_m+1)\lambda_m^{n-k_m}(t),\ \ .....,k_m!\lambda_m(t),(k_m-1)!,\ 0,\ldots & & 0 \end{vmatrix} \qquad (16)$$

differs from zero for any $t\in I$.

Functions $\lambda_i(t),\ i=\overline{1m},\ t\in I$, will be called characteristic functions of matrix $A=A(t)$, $t\in I$.

Consider a scalar function of the scalar argument $f(\lambda)$.

Definition 6. We shall say that function $f(\lambda)$) is defined on the generalized spectrum of spectral matrix $A=A(t)$, $t\in I$, if there are definite and continuous functions on the interval I:

$$f(\lambda_i(t)), f'(\lambda_i(t)),\ ...,\ f^{(k_i-1)}(\lambda_i(t)),\ i=\overline{1m},\ k_1+\cdots+k_m=n,\ t\in I. \qquad (17)$$

Here and hereinafter $f^{(k)}(\lambda)=\dfrac{d^k}{d\lambda^k}f(\lambda), \forall f, k.$

Consider a defining polynomial of n-1 degree with indefinite coefficients

$$P(\lambda)=a_n\lambda^{n-1}+a_{n-1}\lambda^{n-2}+...+a_2\lambda+a_1.$$

Let us demand polynomial $P(\lambda)$ coincide with function $f(\lambda)$ on the generalized spectrum of spectral matrix $A=A(t)$, $t\in I$, i.e. the following condition be fulfilled

$$P(\lambda_i(t))=f(\lambda_i(t)),\ P'(\lambda_i(t))=f'(\lambda_i(t)),\ P^{(k_i-1)}(\lambda_i(t))=f^{(k_i-1)}(\lambda_i(t)),\ \ i=\overline{1m},\ k_1+\cdots+k_m=n,\ t\in I \qquad (18)$$

Condition (18) presents algebraic linear nonhomogeneous n order system with respect to unknown coefficients $a_1,\ \ldots,\ a_n$ of polynomial $P(\lambda)$. It is evident that determinant formed with coefficients of the above-mentioned algebraic system coincides with determinant $\Lambda(A)$ different from zero for any $t\in I$. Consequently, unknown coefficients $a_1, \ldots, a_n$ are defined on the whole interval I unambiguously from condition (18).

Definition 7. Let $f(\lambda)$ be scalar function of the scalar argument λ coinciding with polynomial $P(\lambda)$ on the generalized spectrum of spectral matrix $A=A(t)$, $t\in I$. Let $a_1= a_1(t), \ldots, a_n= a_n(t)$, $t\in I$, be coefficients of (defining) polynomial $P(\lambda)$), defined unambiguously from condition (18).

Then according to definition

$$f(A(t))=P(A(t))=a_1(t)A^{n-1}(t)+\ldots+a_2(t)A(t)+\ldots+a_1(t)E,\ t\in I.$$

The following theorem is evident.

Theorem 4 (Criterion of spectrality). *Let* $A=A(t)=(a_j^i(t))$, $i, j=\overline{1n}$, $t\in I$ *be a triangular matrix* ($a_j^i(t)\equiv 0$ at $i<j$ $i, j=\overline{1n}$ or at $i>j$ $i, j=\overline{1n}$, $t\in I$.) *with diagonal elements* $a_k^k(t)=\mu_k(t)$, $k=\overline{1n}, t\in I$, *that satisfy the following conditions*: $\mu_k(t)=\lambda_i(t), k=\overline{1,k_i}$, $t\in I, i=\overline{1,m}, k_1+\cdots+k_m=n$, $\lambda_i(t)\neq\lambda_j(t)$, $i\neq j, i, j=\overline{1m}$, $t\in I; \Lambda(A)\neq 0$, $t\in I$ (see (16)).

Then $A=A(t)$, $t\in I$, is spectral matrix.

The following theorem is valid.

Theorem 5 (Criterion of regularity). *Let* $A(t)=(a_j^i(t))$ *be an arbitrary n x n matrix, elements of which are definite, continuous and continuously differentiable on the interval I. Then, if* $A(t)$, $t\in I$, *is the spectral matrix permutable with matrix* $\dot{A}(t), t\in I$, *then matrices* $\dot{A}(t)$ and $-\dot{A}(t)$, $t\in I$, *are regular.*

To illustrate the obtained results let us consider an example. Let

$$A=\begin{pmatrix}\alpha,\beta\\ 0,\alpha\end{pmatrix},\ B=\begin{pmatrix}\gamma,0\\ \delta,\gamma\end{pmatrix},$$

where *á=á*(t), *â=â* (t), *ã=ã* (t), *ä* = *ä*(t), $t\in I$, are arbitrary admissible functions.

It is easy to check the permutability of the matrices A, $\dot{A}$ and B, $\dot{B}$. It is obvious that

P(*ë*,*A*)=(*ë*-*á*)^2, *ë*$_1$(t)=á, k_1=2, Ë(A)=–1, $t\in I$;

P(*ë*,B)=(ë- ã)^2, *ë*$_1$(t)= ã, k_1=2, Ë(B)=–1, $t\in I$;

Consequently A and B, $t\in I$ are spectral matrices (Theorem 4) and matrices $\dot{A}, -\dot{A}$, $t\in I$; $\dot{B}, -\dot{B}$, $t\in I$ are regular (Theorem 5).

Let us consider the equation

$$\ddot{X}-(\dot{A}+\dot{B})\dot{X}+(\dot{A}\dot{B}-\ddot{B})X=F,\ t\in I.$$

By virtue of Theorem 1 general solution of this equation has the form:

$$X=e^B\left(C_0+\int e^{-B}\mathrm{Y}dt\right),$$

where

$$\mathrm{Y}=e^A\left(C_1+\int e^{-A}Fdt\right),\quad t\in I;$$

Simple calculations give

$$e^A=e^\alpha\begin{pmatrix}1,\beta\\ 0,1\end{pmatrix},\ \ e^{-A}=e^{-\alpha}\begin{pmatrix}1,-\beta\\ 0,\ 1\end{pmatrix},\quad t\in I,$$

$$e^B=e^\gamma\begin{pmatrix}1,0\\ \delta,1\end{pmatrix},\ \ e^{-B}=e^{-\gamma}\begin{pmatrix}1,\ 0\\ -\delta,\ 1\end{pmatrix},\quad t\in I$$

In conclusion we must notice that the related problems are investigated in [1-5].

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