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Mathematics

On Some Boundary Value Problems with Conditions at Infinity for Nonlinear Differential Systems

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ABSTRACT. For nonlinear differential systems, the boundary value problems are investigated on infinite intervals with conditions at infinity. Optimal, in a certain sense, conditions are found, guaranteeing, respectively, the solvability and unique solvability of these problems, as well as conditional stability of their solutions. *© 2007 Bull. Georg. Natl. Acad. Sci.*

Key words: *nonlinear differential system, boundary value problem with conditions at infinity, solvability, unique solvability, conditional stability.*

In the present paper on the interval *I*, where $I = R_+$ or $I = R$, we consider the nonlinear differential system

$$\frac{dx_i}{dt} = f_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n), \tag{1}$$

where $f_i : I \times R^n \to R$ $(i = 1, \dots, n)$ are functions satisfying the local Carathéodory conditions. In the case $I = R_+$, for the system (1) we investigate the boundary value problem

$$x_i(0) = c_i \quad (i = 1, \dots, m), \quad \limsup_{t \to +\infty} |x_i(t)| < +\infty \quad (i = m+1, \dots, n), \tag{2}$$

and in the case $I = R$ – the boundary value problem

$$\limsup_{t \to -\infty} |x_i(t)| < +\infty \quad (i = 1, \dots, m), \quad \limsup_{t \to +\infty} |x_i(t)| < +\infty \quad (i = m+1, \dots, n). \tag{3}$$

The theorems containing unimprovable sufficient conditions of solvability and unique solvability of the problems (1), (2) and (1), (3) are formulated in Section 1.Under these conditions, solutions of the above-mentioned problems are, generally speaking, unstable in the Ljapunov sense. Therefore there naturally arises the question on their conditional stability in one or another sense. The answer to this question can be found in Section 2. In the same section there are given the notions of $(m, n-m)$**-stability** and **asymptotic** $(m, n-m)$**-stability** of solutions of the system (1), which in a certain sense make more precise the well-known definition of conditional stability (see [1], [2]). Moreover, on the basis of the results obtained in Section 1, the optimal conditions guaranteeing, respectively, the $(m, n-m)$-stability and asymptotic $(m, n-m)$-stability of a trivial solution of the system (1), are found.

Throughout the paper the use will be made of the following notation:

$$R =]-\infty,+\infty[, \quad R_+ = [0,+\infty[, \quad R_- =]-\infty,0];$$

R^n is the n-dimensional real Euclidian space; $x=(x_i)_{i=1}^n \in R^n$ is the vector with components x_i $(i=1,\dots,n)$; $R^n(\gamma)=\{x=(x_i)_{i=1}^n \in R^n : |x_1| \le \gamma,\dots,|x_n| \le \gamma\}$; δ_{ik} is Kronecker's symbol, i.e.,

$$\delta_{ik} = \begin{cases} 1 & \text{for } i = k \\ 0 & \text{for } i \ne k \end{cases};$$

$X=(x_{ik})_{i,k=1}^n$ is the $n\times n$-matrix with components $x_{ik} \in R$ $(i,k=1,\dots,n)$; $r(X)$ is the spectral radius of X; A_s is the set of asymptotically stable, quasi-nonnegative $n\times n$-matrices, i.e. $H=(h_{ik})_{i,k=1}^n \in A_s$ if and only if $h_{ik} \ge 0$ for $i \ne k$ and real parts of eigenvalues of H are negative; $\widetilde{C}_{loc}(I)$ is the space of functions $x: I \to R$, absolutely continuous on every compact interval containing in I; $L_{loc}(I)$ is the space of functions $x: I \to R$, Lebesgue integrable on every compact interval containing in I; $L^\infty(I)$ is the space of essentially bounded functions $x: I \to R$ with the norm

$$\|x\|_{L^\infty} = \operatorname{ess\,sup}\{|x(t)| : t \in I\};$$

$\mathcal{K}_{loc}(I\times D)$, where $D \subset R^n$, is the set of functions $f: I\times D \to R$, satisfying the local Carathéodory conditions.

1. Existence and Uniqueness Theorems

Everywhere below, when we deal with the problems (1), (2) and (1), (3), we assume, respectively, that

$$f_i \in \mathcal{K}_{loc}(R_+ \times R^n) \quad (i=1,\dots,n)$$

and

$$f_i \in \mathcal{K}_{loc}(R \times R^n) \quad (i=1,\dots,n).$$

By a solution of the system (1), defined on the interval I, is understood the vector function $(x_i)_{i=1}^n : I \to R^n$ with components $x_i \in \widetilde{C}_{loc}(I)$ $(i=1,\dots,n)$, which almost everywhere on I satisfies this system.

A solution $(x_i)_{i=1}^n$ of the system (1), defined on R_+ (defined on R) and satisfying the boundary conditions (2) (the boundary conditions (3)), is called a solution of the problem (1), (2) (of the problem (1), (3)).

Theorem 1.1. *Let there exist nonnegative functions* $g_i \in \mathcal{K}_{loc}(R_+ \times R^n)$ $(i=1,\dots,n)$, $h \in L^\infty(R_+)$ *and a constant matrix* $H=(h_{ik})_{i,k=1}^n$ *such that*

$$H \in A_s, \tag{4}$$

and on the set $R_+ \times R^n$ the inequalities

$$\sigma_i f_i(t,x_1,\dots,x_n)\operatorname{sgn}(x_i) \le g_i(t,x_1,\dots,x_n)\left(\sum_{k=1}^n h_{ik}|x_k| + h(t)\right) \quad (i=1,\dots,n), \tag{5}$$

where $\sigma_1=\dots=\sigma_m=1$ *and* $\sigma_{m+1}=\dots=\sigma_n=-1$, *are satisfied. Then for arbitrary* $c_i \in R$ $(i=1,\dots,m)$ *the problem* (1), (2) *has at least one solution, and every solution of that problem is bounded on* R_+.

It is known (see [4], Theorem 1.18) that the quasi-nonnegative matrix $H=(h_{ik})_{i,k=1}^n$ satisfies the condition (4) iff

$$h_{ii} < 0 \quad (i=1,\dots,n) \quad \text{and} \quad r(H_0) < 1, \tag{6}$$

where

$$H_0 = \left((1-\delta_{ik}) \frac{h_{ik}}{|h_{ii}|} \right)_{i,k=1}^{n}.$$

Thus the condition (4) in Theorem 1.1 can be replaced by the equivalent condition (6).

Note that the condition (4) both in Theorem 1.1 and in other theorems below is unimprovable and it cannot be weakened. In particular, the condition (4) cannot be replaced by the condition

$$h_{ii} < 0 \quad (i = 1,\dots,n), \quad r(H_0) \le 1.$$

Corollary 1.1. *Let the conditions of Theorem* 1.1 *be fulfilled and*

$$\int_0^{+\infty} p_i(s)\,ds = +\infty \quad (i = m+1,\dots,n), \tag{7}$$

where

$$p_i(t) = \inf\left\{ g_i(t, x_1,\dots,x_n) : (x_k)_{k=1}^n \in R^n \right\}. \tag{8}$$

Then every solution of the problem (1), (2) *admits the estimate*

$$\sum_{k=1}^{m} |x_k(t)| \le \rho \left(\sum_{k=1}^{m} |c_k| + \|h\|_{L^\infty} \right) \quad for \quad t \in R_+, \tag{9}$$

where ρ *is a positive constant, depending only on H.*

From the estimates (9) it, in particular, follows that if the conditions of Corollary 1.1 are fulfilled, then an arbitrary solution of the system (1), satisfying the conditions

$$x_i(0) = c_i \quad (i = 1,\dots,m) \quad \text{and} \quad \sum_{k=1}^{n} |x_k(0)| > \rho \left(\sum_{k=1}^{m} |c_k| + \|h\|_{L^\infty} \right),$$

is either unbounded, or blowing-up.

Corollary 1.2. *Let the conditions of Theorem* 1.1 *be fulfilled and*

$$\lim_{t\to+\infty} h(t) = 0, \qquad \int_0^{+\infty} p_i(s)\,ds = +\infty \quad (i = 1,\dots,n), \tag{10}$$

where each p_i *is the function given by the equality* (8). *Then an arbitrary solution of the problem* (1), (2) *satisfies the equalities*

$$\lim_{t\to+\infty} x_i(t) = 0 \quad (i = 1,\dots,n). \tag{11}$$

Theorem 1.2. *Let there exist nonnegative functions* $p_i \in L_{loc}(R_+)$ $(i = 1,\dots,n)$, $h \in L^\infty(R_+)$, *and a constant matrix* $H = (h_{ik})_{i,k=1}^n \in A_s$ *such that, respectively, on* $R_+ \times R^n$ *and* R_+ *the conditions*

$$\sigma_i \left(f_i(t, x_1,\dots,x_n) - f_i(t, y_1,\dots,y_n) \right) \operatorname{sgn}(x_i - y_i) \le p_i(t) \sum_{k=1}^{n} h_{ik} |x_k - y_k| \quad (i = 1,\dots,n), \tag{12}$$

$$|f_i(t,0,\dots,0)| \le h(t)\, p_i(t) \quad (i = 1,\dots,n), \tag{13}$$

where $\sigma_1 = \dots = \sigma_m = 1$, $\sigma_{m+1} = \dots = \sigma_n = -1$, *are satisfied. If, moreover, the equalities* (7) (*the equalities* (10)) *hold, then for arbitrary* $c_i \in R$ $(i = 1,\dots,m)$ *the problem* (1), (2) *has a unique solution satisfying the condition* (9) (*the conditions* (9) *and* (11)), *where* ρ *is a positive constant, depending only on H.*

Let us now consider the problem (1),(3). The following theorem is valid.

Theorem 1.3. *Let there exist nonnegative functions* $g_i \in \mathcal{K}_{loc}(R \times R^n)$ $(i = 1,\dots,n)$ *and a constant matrix* $H = (h_{ik})_{i,k=1}^n \in A_s$ *such that on the set* $R \times R^n$ *the inequalities* (5) *are satisfied, where* $\sigma_1 = \cdots = \sigma_m = 1$ *and* $\sigma_{m+1} = \cdots = \sigma_n = -1$. *Then the problem* (1), (3) *has at least one solution, and every solution of that problem is bounded on R.*

Unlike Theorems 11.2_1 and 11.2_2 proven in [3], Theorems 1.2 and 1.3 cover the cases in which the right-hand sides of the system (1) are functions, rapidly increasing with respect to the phase variables.

Corollary 1.3. *Let the conditions of Theorem* 1.3 *be fulfilled and*

$$\int_{-\infty}^{0} p_i(s)ds = +\infty \quad (i = 1,\dots,m), \qquad \int_{0}^{+\infty} p_i(s)ds = +\infty \quad (i = m+1,\dots,n), \tag{14}$$

where each p_i *is the function given by the equality* (8). *Then every solution of the problem* (1), (3) *admits the estimates*

$$|x_i(t)| \le \rho \| h \|_{L^\infty} \quad \text{for} \quad t \in R \quad (i = 1,\dots,n), \tag{15}$$

where ρ *is a positive constant, depending only on H. If, however, instead of* (14) *are satisfied the conditions*

$$\lim_{t\to-\infty} h_i(t) = \lim_{t\to+\infty} h_i(t) = 0, \qquad \int_{-\infty}^{0} p_i(s)ds = \int_{0}^{\infty} p_i(s)ds = +\infty \quad (i = 1,\dots,n), \tag{16}$$

then every solution of the problem (1), (3) *along with* (15) *satisfies the conditions*

$$\lim_{t\to-\infty} x_i(t) = \lim_{t\to+\infty} x_i(t) = 0 \quad (i = 1,\dots,n). \tag{17}$$

Theorem 1.4. *Let there exist nonnegative functions* $p_i \in L_{loc}(R)$ $(i = 1,\dots,n)$, $h \in L^\infty(R)$ *and a constant matrix* $H = (h_{ik})_{i,k=1}^n \in A_s$ *such that, respectively, on* $R \times R^n$ *and R the conditions* (12) *and* (13), *where* $\sigma_1 = \cdots = \sigma_m = 1$, $\sigma_{m+1} = \cdots = \sigma_n = -1$, *are satisfied. If, moreover, the equalities* (14) (*the equalities* (16)) *are fulfilled, then the problem* (1), (3) *has a unique solution satisfying the condition* (15) (*the conditions* (15) *and* (17)), *where* ρ *is a positive constant, depending only on H.*

2. Theorems on the Conditional Stability

Let us consider first the case where for some $\gamma > 0$,

$$f_i \in \mathcal{K}_{loc}(R_+ \times R^n(\gamma)) \quad (i = 1,\dots,n),$$

and the system (1) on R_+ has a trivial solution, i.e. $f_i(t,0,\dots,0) \equiv 0$ $(i = 1,\dots,n)$.

For the system (1) we have to consider the following boundary value problems

$$x_i(a) = c_i \quad (i = 1,\dots,m), \qquad x_i(b) = c_i \quad (i = m+1,\dots,n) \tag{18}$$

and

$$x_i(a) = c_i \quad (i = 1,\dots,m), \qquad \limsup_{t\to+\infty} |x_i(t)| < +\infty \quad (i = m+1,\dots,n), \tag{19}$$

where $m \in \{1,\dots,n-1\}$.

Introduce the following

Definition 2.1. A trivial solution of the system (1) is said to be $(m, n-m)$-stable on R_+ if for any $\varepsilon \in \,]0,\gamma[$ there exists $\delta \in]0,\varepsilon[$ such that:

(i) for arbitrary $a \in R_+$, $b \in]a,+\infty[$, and $c_i \in [-\delta,\delta]$ $(i=1,\dots,n)$, the problem (1), (18) has at least one solution, and every solution of that problem on $[a,b]$ satisfies the inequality

$$\sum_{i=1}^{n} |x_i(t)| < \varepsilon ; \tag{20}$$

(ii) for arbitrary $a \in R_+$ and $c_i \in [-\delta,\delta]$ $(i=1,\dots,m)$, the problem (1), (19) has at least one solution, and every solution of that problem satisfies on $[a,+\infty[$ the inequality (20).

Definition 2.2. A trivial solution of the system (1) is said to be asymptotically $(m,n-m)$-stable on R_+ if it is $(m,n-m)$-stable on R_+ and there exists $\delta_0 \in]0,\gamma[$ such that for arbitrary $a \in R_+$ and $c_i \in [-\delta_0,\delta_0]$ $(i=1,\dots,m)$ every solution of the problem (1), (19) satisfies the equalities (11).

Theorem 2.1. *Let on* $R_+ \times R^n(\gamma)$ *the inequalities*

$$\sigma_i f_i(t,x_1,\dots,x_n)\,\mathrm{sgn}(x_i) \le g_i(t,x_1,\dots,x_n) \sum_{k=1}^{n} h_{ik} |x_k| \quad (i=1,\dots,n) \tag{21}$$

be fulfilled, where $\sigma_1 = \dots = \sigma_m = 1$, $\sigma_{m+1} = \dots = \sigma_n = -1$, $(h_{ik})_{i,k=1}^{n} \in A_s$, *and* $g_i \in \mathcal{K}_{loc}(R_+ \times R^n(\gamma))$ $(i=1,\dots,n)$ *are nonnegative functions. If, moreover, the conditions* (7), *where each* p_i *is the function given by the equality* (8), *are fulfilled, then a trivial solution of the system* (1) *is* $(m,n-m)$*-stable on* R_+. *If, however, instead of* (7) *we have*

$$\int_0^{+\infty} p_i(s)ds = +\infty \quad (i=1,\dots,n), \tag{22}$$

then a trivial solution of the system (1) *is asymptotically* $(m,n-m)$*-stable on* R_+.

As an example, consider the case when

$$f_i(t,x_1,\dots,x_n) = -p_i(t)x_i \quad (i=1,\dots,m), \qquad f_i(t,x_1,\dots,x_n) = p_i(t)x_i \quad (i=m+1,\dots,n), \tag{23}$$

and $p_i \in L_{loc}(R_+)$ $(i=1,\dots,n)$ are nonnegative functions. Then for the system (1) to be $(m,n-m)$-stable (asymptotically $(m,n-m)$-stable), it is necessary and sufficient that the conditions (7) (the conditions (22)) be fulfilled. Consequently, conditions (7), (conditions (22)) in Theorem 2.1 are optimal, and they cannot be weakened.

Consider the linear differential system

$$\frac{dx_i}{dt} = \sum_{k=1}^{n} p_{ik}(t)x_k \quad (i=1,\dots,n) \tag{24}$$

with coefficients $p_{ik} \in L_{loc}(R_+)$ $(i,k=1,\dots,n)$. We call this system $(m,n-m)$-stable (asymptotically $(m,n-m)$-stable) on R_+ if its trivial solution is $(m,n-m)$-stable (asymptotically $(m,n-m)$-stable) on R_+.

From Theorem 2.1 it follows

Corollary 2.1. *Let on* R_+ *the conditions*

$$p_{ii}(t) = \sigma_i \ell_{ii} p_i(t), \quad |p_{ik}(t)| \le \ell_{ik} |p_i(t)| \quad (i,k=1,\dots,n; \;\; i \ne k) \tag{25}$$

be satisfied, where $\sigma_1 = \dots = \sigma_m = 1$, $\sigma_{m+1} = \dots = \sigma_n = -1$, *and* $(h_{ik})_{i,k=1}^{n} \in A_s$. *If, moreover, the equalities* (7) (*the equalities* (22)) *are fulfilled, then the system* (24) *is* $(m,n-m)$*-stable* (*asymptotically* $(m,n-m)$*-stable*) *on* R_+.

It can be easily seen that under the conditions of Corollary 2.1 the bounded solutions of the system (23) form an m-dimensional linear space. Moreover, if the equalities (22) are fulfilled, then all solutions from the above-mentioned space are vanishing at $+\infty$.

Let us now consider the case, where for some $\gamma > 0$,

$$f_i \in \mathcal{K}_{loc}\left(R \times R^n(\gamma)\right) \quad (i = 1,\dots,n)$$

and $f_i(t,0,\dots,0) \equiv 0 \quad (i = 1,\dots,n)$. In this case, for the system (1) along with the problems (18) and (19) we have to consider also the problem

$$x_i(a) = c_{i-m} \quad (i = m+1,\dots,n), \qquad \limsup_{t\to-\infty} |x_i(t)| < +\infty \quad (i = 1,\dots,m). \tag{26}$$

Definition 2.3. A trivial solution of the system (1) is said to be $(m, n-m)$-stable on R if for any $\varepsilon \in]0,\gamma[$ there exists $\delta \in]0,\varepsilon[$ such that:

(i) for arbitrary $a \in R$, $b \in]a,+\infty[$, $c_i \in [-\delta,\delta]$ $(i = 1,\dots,n)$, the problem (1), (18) has at least one solution, and every solution of that problem satisfies on $[a,b]$ the inequality (20);

(ii) for arbitrary $a \in R$ and $c_i \in [-\delta,\delta]$ $(i = 1,\dots,m)$ (for arbitrary $a \in R$ and $c_i \in [-\delta,\delta]$ $(i = 1,\dots,n-m)$), the problem (1), (19) (the problem (1), (26)) has at least one solution, and every solution of that problem satisfies on $[a,+\infty[$ (on $]-\infty,a]$) the inequality (20).

Definition 2.4. A trivial solution of the system (1) is said to be asymptotically $(m,n-m)$-stable on R if it is $(m,n-m)$-stable on R, and there exists $\delta_0 \in]0,\omega[$ such that for arbitrary $a \in R$, $c_i \in [-\delta_0,\delta_0]$ $(i = 1,\dots,m)$ (for arbitrary $a \in R$ and $c_i \in [-\delta_0,\delta_0]$ $(i = 1,\dots,n-m)$) every solution of the problem (1), (19) (of the problem (1), (26)) satisfies the equalities

$$\lim_{t\to+\infty} x_i(t) = 0 \quad (i = 1,\dots,n); \qquad \left(\lim_{t\to-\infty} x_i(t) = 0 \quad (i = 1,\dots,n)\right).$$

Theorem 2.2. *Let on* $R \times R^n(\gamma)$ *the inequalities* (21) *be satisfied, where* $\sigma_1 = \dots = \sigma_m = 1$, $\sigma_{m+1} = \dots = \sigma_n = -1$, $(h_{ik})_{i,k=1}^n \in A_s$, *and* $g_i \in \mathcal{K}_{loc}\left(R \times R^n(\gamma)\right)$ $(i = 1,\dots,n)$ *are nonnegative functions. If, moreover, the conditions* (14) *are fulfilled, where each* p_i *is the function given by the equality* (8), *then a trivial solution of the system* (1) *is* $(m,n-m)$*-stable on R. If, however, instead of* (14) *we have*

$$\int_{-\infty}^{0} p_i(s)\,ds = \int_{0}^{+\infty} p_i(s)\,ds = +\infty \quad (i = 1,\dots,n), \tag{27}$$

then a trivial solution of the system (1) *is asymptotically* $(m,n-m)$*-stable on R.*

If the right-hand sides of the system (1) are of the form (23), where $p_i \in L_{loc}(R)$ $(i = 1,\dots,n)$ are nonnegative functions, then for a trivial solution of the system (1) to be $(m,n-m)$-stable (asymptotically $(m,n-m)$-stable) on R, it is necessary and sufficient that the conditions (14) (the conditions (27)) be fulfilled. Consequently, the conditions of Theorem 2.2 are in a certain sense unimprovable.

For the linear differential system (24) with coefficients $p_{ik} \in L_{loc}(R)$ $(i,k = 1,\dots,n)$ from Theorem 2.2 it follows

Corollary 2.2. *Let on R the conditions* (25) *be satisfied, where* $\sigma_1 = \dots = \sigma_m = 1$, $\sigma_{m+1} = \dots = \sigma_n = -1$ *and* $(h_{ik})_{i,k=1}^n \in A_s$. *If, moreover, the equalities* (14) (*the equalities* (27)) *are fulfilled, then the system* (24) *is* $(m,n-m)$*-stable* (*asymptotically* $(m,n-m)$*-stable*) *on R.*

It is clear that under the conditions of Corollary 2.2 the system (24) has no nontrivial bounded solution on R, and a set of bounded on R_+ (bounded on R_-) solutions of that system forms an m-dimensional (an $(n$-$m)$-dimensional) linear space.

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მათემატიკა

უსასრულობაში პირობებიანი ზოგიერთი სასაზღვრო ამოცანის შესახებ არაწრფივი დიფერენციალური სისტემებისათვის

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ნაპოვნია (1), (2) და (1), (3) სასაზღვრო ამოცანების ამოხსნადობისა და ცალსახად ამოხსნადობის არაგაუმჯობესებადი საკმარისი პირობები და ამ შედეგებზე დაყრდნობით გამოკვლეულია (1) სისტემის ამონახსნების პირობითი მდგრადობის საკითხი. სახელდობრ, შემოღებულია $(m, n-m)$-მდგრადობისა და ასიმპტოტურად $(m, n-m)$-მდგრადობის ცნებები და დადგენილია გარკვეული აზრით ოპტიმალური პირობები, რომლებიც სათანადოდ უზრუნველყოფენ (1) სისტემის ტრივიალური ამონახსნის $(m, n-m)$-მდგრადობასა და ასიმპტოტურად $(m, n-m)$-მდგრადობას.

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