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On Dynamical Hierarchical Models of Multistructures

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ABSTRACT. In the present paper a hierarchy of dynamical models of multistructures consisting of parts with different shapes is constructed. The existence and uniqueness of solutions of the obtained initial-boundary value problems defined on the union of one-, two- and three-dimensional domains is proved and the relation of the reduced problems to the original three-dimensional problem is studied. © *2007 Bull. Georg. Natl. Acad. Sci.*

Key words: *elastic multistructures, hierarchical modeling.*

Investigation of mathematical models of elastic bodies consisting of several parts is important both from the theoretical and from practical points of view, because such type of multistructures is widely used in various engineering constructions [1,2]. The present paper is devoted to the construction and investigation of a dynamical hierarchical model of multistructure, which consists of a plate, rod and three-dimensional body. Applying approximation by single and double Fourier-Legendre series of the components of displacement vector function, the three-dimensional models of plate and rod are reduced to two-dimensional and one-dimensional ones, respectively [3-7]. The obtained initial boundary value problem is defined on the union of three-, two- and one-dimensional domains and the existence and uniqueness of its solution are investigated. Moreover, we prove that the sequence of vector functions of three space variables converges to the solution of the original three-dimensional problem and the rate of convergence is estimated.

Let $\Omega \subset \mathbf{R}^p, p \geq 1$ be a bounded domain with Lipschitz boundary. $L^2(\Omega)$ denotes the space of square-integrable functions in Ω in the Lebesgue sense. $W^{k,2}(\Omega) = H^k(\Omega)$, $k \geq 1$, is the Sobolev space of order k, i.e. the subspace of $L^2(\Omega)$ of functions whose derivatives in the sense of distributions up to the order k belong to $L^2(\Omega)$, $\mathbf{H}^k(\Omega) = \left[H^k(\Omega)\right]^3$, $\mathbf{L}^2(\Omega) = \left[L^2(\Omega)\right]^3$. $C^0([0,T];X)$ and $L^2(0,T;X)$ are the spaces of continuous and square integrable vector functions with values in Banach space X. Assume that the indices i, j, p, q vary in the set $\{1,2,3\}$ and the summation convention with respect to the repeated indices is used.

Let us consider an elastic body Ω with initial configuration $\overline{\Omega} = \overline{\Omega^3} \cup \overline{\Omega^h} \cup \overline{\Omega^{h_1 h_2}}$, which consists of three-dimensional substructure $\Omega^3 = (-d_1^1, -d_2^1) \times (-d_1^2, d_2^2) \times (-d_1^3, d_2^3)$, elastic plate $\Omega^h = \omega \times (-h, h)$, $\omega = (-l_1^1, l_2^1) \times (-l_1^2, l_2^2)$, and elastic rod $\Omega^{h_1 h_2} = (-h_1, h_1) \times (-h_2, h_2) \times I$, $I = (-h, d_2)$, where the parameters characterizing the multistructure are positive and satisfy the following conditions:

$$\min\{d_2, d_1^3, d_2^3\} > h\,,\ \min\{l_2^1, d_1^1\} > h_1\,,\ \min\{l_1^2, l_2^2\} > h_2\,,$$

$$d_2^2 \geq l_2^2,\ d_1^2 \geq l_1^2,\ d_1^1 > l_1^1 > d_2^1.$$

The body Ω is clamped along the part $\Gamma_0 = (-d_1^1, -d_2^1) \times (-d_1^2, d_2^2) \times \{d_2^3\}$ of its boundary $\Gamma = \partial\Omega$ and surface force with density $\mathbf{g} = (g_i): \Gamma \setminus \Gamma_0 \times (0,T) \to \mathbf{R}^3 \times (0,T)$ is acting on the remaining part of the boundary. We denote the applied body force density by $\mathbf{f} = (f_i): \Omega \times (0,T) \to \mathbf{R}^3 \times (0,T)$.

We consider a dynamical problem for an elastic multistructure which consists of general anisotropic nonhomogeneous material with components a_{ijpq} of the linearized elasticity tensor. The dynamical problem admits the following variational formulation: find $\mathbf{u} \in C^0([0,T]; V(\Omega))$, $\mathbf{u}' \in C^0([0,T]; \mathbf{L}^2(\Omega))$, which in the sense of distributions in $(0,T)$ satisfies the equation

$$\frac{d}{dt}(\mathbf{u}'(.), \mathbf{v})_{\mathbf{L}^2(\Omega)} + A(\mathbf{u}(.), \mathbf{v}) = L(\mathbf{v}), \quad \forall \mathbf{v} \in V(\Omega), \tag{1}$$

and the following initial conditions

$$\mathbf{u}(0) = \boldsymbol{\varphi}, \quad \mathbf{u}'(0) = \boldsymbol{\psi}, \tag{2}$$

where $\boldsymbol{\varphi} \in V(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega);\ \mathbf{v} = 0 \ \text{ on } \Gamma_0\}$, $\boldsymbol{\psi} \in \mathbf{L}^2(\Omega)$ are the initial displacement and velocity vector functions,

$$A(\mathbf{v}, \mathbf{v}_1) = \sum_{i,j,p,q=1}^{3} a_{ijpq}(x) e_{pq}(\mathbf{v}) e_{ij}(\mathbf{v}_1) dx,$$

$$L(v) = \sum_{i=1}^{3} \int_{\Omega} f_i v_i dx + \sum_{i=1}^{3} \int_{\Gamma \setminus \Gamma_0} g_i v_i d\Gamma, \quad e_{pq}(\mathbf{v}) = \frac{1}{2}\left(\frac{\partial v_p}{\partial x_q} + \frac{\partial v_q}{\partial x_p}\right), \quad \forall \mathbf{v}, \mathbf{v}_1 \in V(\Omega).$$

Each vector function $\mathbf{v} \in V(\Omega)$ can be represented as a triple of vector functions $\mathbf{v}^{bd}$, $\mathbf{v}^{pl}$ and $\mathbf{v}^{rd}$, which are restrictions of $\mathbf{v}$ on the sets Ω^3, Ω^h and $\Omega^{h_1 h_2}$, respectively. Therefore, $V(\Omega)$ can be considered as a space of triples $(\mathbf{v}^{bd}, \mathbf{v}^{pl}, \mathbf{v}^{rd}) \in \mathbf{H}^1(\Omega^3) \times \mathbf{H}^1(\Omega^h) \times \mathbf{H}^1(\Omega^{h_1 h_2})$, where $\mathbf{v}^{bd} = 0$ on Γ_0, $\mathbf{v}^{bd} = \mathbf{v}^{pl}$ in $\Omega^3 \cap \Omega^h$, $\mathbf{v}^{pl} = \mathbf{v}^{rd}$ in $\Omega^h \cap \Omega^{h_1 h_2}$.

To obtain a dynamical model for the multistructure we consider the subspaces $V_{\mathbf{N}}(\Omega) \subset V(\Omega)$ and $\mathbf{L}^2_{\mathbf{N}}(\Omega) \subset \mathbf{L}^2(\Omega)$, $\mathbf{N} = (N_1, N_2, N_3)$, which consist of triples $\mathbf{v}_{\mathbf{N}} = (\mathbf{v}^{bd}_{\mathbf{N}}, \mathbf{v}^{pl}_{N_3}, \mathbf{v}^{rd}_{N_1 N_2})$, where $\mathbf{v}^{pl}_{N_3}$ is a polynomial of degree N_3 with respect to the variable x_3, $\mathbf{v}^{rd}_{N_1 N_2}$ is a polynomial of degrees N_1 and N_2 with respect to the variables x_1 and x_2, and on the sets $\Omega^3 \cap \Omega^h$, $\Omega^h \cap \Omega^{h_1 h_2}$ the compatibility conditions are fulfilled. Using these subspaces from the original problem we obtain problems defined on the union of three, two and one-dimensional domains: find a vector function $\overline{\mathbf{w}}_{\mathbf{N}} \in C^0([0,T]; \overline{V}_{\mathbf{N}}(\Omega^*))$, $\overline{\mathbf{w}}'_{\mathbf{N}} \in C^0([0,T]; \overline{H}_{\mathbf{N}}(\Omega^*))$, $\Omega^* = \Omega^3 \times \omega \times I$, such that

$$\frac{d}{dt} Q_{\mathbf{N}}(\overline{\mathbf{w}}'_{\mathbf{N}}(.), \overline{\mathbf{v}}_{\mathbf{N}}) + A_{\mathbf{N}}(\overline{\mathbf{w}}_{\mathbf{N}}(.), \overline{\mathbf{v}}_{\mathbf{N}}) = L_{\mathbf{N}}(\overline{\mathbf{v}}_{\mathbf{N}}), \quad \forall \overline{\mathbf{v}}_{\mathbf{N}} \in \overline{V}_{\mathbf{N}}(\Omega^*), \tag{3}$$

$$\overline{\mathbf{w}}_{\mathbf{N}}(0) = \overline{\boldsymbol{\varphi}}_{\mathbf{N}}, \quad \overline{\mathbf{w}}'_{\mathbf{N}}(0) = \overline{\boldsymbol{\psi}}_{\mathbf{N}}, \tag{4}$$

where $\overline{\boldsymbol{\varphi}}_{\mathbf{N}} \in \overline{V}_{\mathbf{N}}(\Omega^*)$, $\overline{\boldsymbol{\psi}}_{\mathbf{N}} \in \overline{H}_{\mathbf{N}}(\Omega^*)$, $\overline{V}_{\mathbf{N}}(\Omega^*) = \{\overline{\mathbf{v}}_{\mathbf{N}} = (\mathbf{v}^{bd}_{\mathbf{N}}, \overline{\mathbf{v}}^{pl}_{N_3}, \overline{\mathbf{v}}^{rd}_{N_1 N_2});\ \mathbf{v}^{bd}_{\mathbf{N}} \in \mathbf{H}^1(\Omega^3)$, $\mathbf{v}^{bd}_{\mathbf{N}} = 0$ on Γ_0, $\overline{\mathbf{v}}^{pl}_{N_3} = (\overset{k_3}{\mathbf{v}}{}^{pl}_{N_3})$, $\overset{k_3}{\mathbf{v}}{}^{pl}_{N_3} \in \mathbf{H}^1(\omega)$, $\overline{\mathbf{v}}^{rd}_{N_1 N_2} = (\overset{k_1 k_2}{\mathbf{v}}{}^{rd}_{N_1 N_2})$, $\overset{k_1 k_2}{\mathbf{v}}{}^{rd}_{N_1 N_2} \in \mathbf{H}^1(I)$, $k_i = \overline{0, N_i}$, $i = \overline{1,3}$, $\mathbf{v}^{bd}_{\mathbf{N}} = \mathbf{v}^{pl}_{N_3}$ in $\Omega^3 \cap \Omega^h$, $\mathbf{v}^{pl}_{N_3} = \mathbf{v}^{rd}_{N_1 N_2}$ in $\Omega^h \cap \Omega^{h_1 h_2}\}$, and similarly we define the space $\overline{H}_{\mathbf{N}}(\Omega^*)$ with L^2 instead of H^1 and without any conditions on Γ_0, $Q_{\mathrm{N}}(\overline{\mathbf{v}}^1_{\mathrm{N}}, \overline{\mathbf{v}}_{\mathrm{N}}) = (\mathbf{v}^1_{\mathrm{N}}, \mathbf{v}_{\mathrm{N}})_{L^2(\Omega)}$, $A_{\mathrm{N}}(\overline{\mathbf{v}}^1_{\mathrm{N}}, \overline{\mathbf{v}}_{\mathrm{N}}) = A(\mathbf{v}^1_{\mathrm{N}}, \mathbf{v}_{\mathrm{N}})$, $L_{\mathrm{N}}(\overline{\mathbf{v}}_{\mathrm{N}}) = L(\mathbf{v}_{\mathrm{N}})$, for all $\overline{\mathbf{v}}^1_{\mathrm{N}}, \overline{\mathbf{v}}_{\mathrm{N}} \in \overline{V}_{\mathrm{N}}(\Omega^*)$, which correspond to $\mathbf{v}^1_{\mathrm{N}}, \mathbf{v}_{\mathrm{N}} \in V_{\mathrm{N}}(\Omega)$.

For the constructed hierarchical model the following theorem is proved.

Theorem. *If* $\mathbf{f} \in \mathbf{L}^2(\Omega \times (0,T))$, $\mathbf{g}, \mathbf{g}' \in L^2(0,T; \mathbf{L}^{4/3}(\Gamma \setminus \Gamma_0))$, $a_{ijpq} \in L^\infty(\Omega)$, *and there exists* $\alpha = const > 0$ *such that for all* $\varepsilon_{ij} \in \mathbf{R}$, $\varepsilon_{ij} = \varepsilon_{ji}$,

$$a_{ijpq}(x) = a_{jipq}(x) = a_{pqij}(x), \qquad \sum_{\mathrm{i,j,p,q}=1}^{3} a_{ijpq}(x)\varepsilon_{ij}\varepsilon_{pq} \geq \alpha \sum_{i,j=1}^{3} \varepsilon_{ij}\varepsilon_{ij}, \quad \forall x \in \Omega,$$

then the problem (3), (4) has a unique solution $\overline{\mathbf{w}}_{\mathrm{N}}(t)$. *The sequence of corresponding vector functions of three space variables* $\{\mathbf{w}_{\mathrm{N}}(t)\}$ *tends to the solution* $\mathbf{u}(t)$ *of the three-dimensional problem (1), (2), whenever vector functions* $\boldsymbol{\varphi}_{\mathrm{N}} \in V_{\mathrm{N}}(\Omega)$, $\boldsymbol{\psi}_{\mathrm{N}} \in \mathbf{L}^2_{\mathrm{N}}(\Omega)$ *corresponding to the initial conditions* $\overline{\boldsymbol{\varphi}}_{\mathrm{N}}$, $\overline{\boldsymbol{\psi}}_{\mathrm{N}}$ *of the reduced problem tend to* $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ *in the spaces* $V(\Omega)$ *and* $\mathbf{L}^2(\Omega)$, *respectively,*

$$\begin{matrix} \mathbf{w}_{\mathrm{N}}(t) \to \mathbf{u}(t) \quad \text{in } V(\Omega), \\ \mathbf{w}'_{\mathrm{N}}(t) \to \mathbf{u}'(t) \quad \text{in } \mathbf{L}^2(\Omega), \end{matrix} \quad as \ N_{\min} = \min\{N_1, N_2, N_3\} \to \infty, \forall t \in [0,T].$$

Moreover, if $\mathbf{u}$ *satisfies additional regularity conditions with respect to the space variables* $d^p\mathbf{u}^{bd}/dt^p \in L^2(0,T; \mathbf{H}^2(\Omega^3))$, $d^p\mathbf{u}^{pl}/dt^p \in L^2(0,T; \mathbf{H}^{s_p}(\Omega^h))$, $d^p\mathbf{u}^{rd}/dt^p \in L^2(0,T; \mathbf{H}^{s_p}(\Omega^{h_1h_2}))$, $s_p \in \mathbf{N}$, $p = 0,1,2$, $s_0 \geq s_1 \geq s_2 \geq 2$, *then for suitable* $\overline{\boldsymbol{\varphi}}_{\mathrm{N}}$, $\overline{\boldsymbol{\psi}}_{\mathrm{N}}$ *the following estimate is valid:*

$$\|\mathbf{u}'(t) - \mathbf{w}'_{\mathrm{N}}(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}(t) - \mathbf{w}_{\mathrm{N}}(t)\|^2_{\mathbf{H}^1(\Omega)} \leq \frac{1}{N_{\min}^{2s-3}} o(T, \Omega, N_1, N_2, N_3), \quad \forall t \in [0,T],$$

where $s = \min\{s_1, s_2\}$ *and* $o(T, \Omega, N_1, N_2, N_3) \to 0$, *as* $N_{\min} \to \infty$.

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