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Physics

Numerical Calculation of Distribution of Induced Charge Density on Planar Confined Surfaces

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ABSTRACT. The calculation method of distribution of induced charge density on planar surfaces, including fractal structures of Sierpinski carpet type, is proposed. The calculation scheme is based on the fact that simply connected conducting surface of arbitrary geometry is an equipotential surface. *© 2007 Bull. Georg. Natl. Acad. Sci.*

Key words: *electrostatics, charge distribution, induced charges.*

One of the problems of electrostatics, namely, that of calculation of the distribution of induced charge density (ICD) on conducting surfaces is very urgent for many fields of physics and technology. The charge inducing the distribution may be both point charge and a charged conductor of finite size. At present induced charges play an important role in solving problems in the field of nanotechnology [1], in the study of piezoelectric and electrokinetic phenomena [2,3], as well as of the properties of plasma near the walls of thermonuclear installations [4].

Knowledge of the distribution of induced charges allows calculation of the spatial distribution of corresponding potentials and electric fields. For example, it is important in the research of discharging processes and for obtaining real recommendations for protection from lightning.

The problem is sufficiently complicated for finite electrodynamic structures of arbitrary geometry and frequently it is impossible to solve it analytically. Therefore, solution of the problem calls for numerical methods. A fairly simple algorithm of calculation of induced charges density is given in the paper, elaborated by us for various types of similar problems.

Algorithm of calculation of induced charge density on plane surfaces of finite size

The calculation scheme of determination of ICD on a conducting structure is based on the main postulate of electrostatics – each point of conducting surface has one and the same potential. In other words, a conducting surface is an equipotential surface. If we put a charge at a distance from the conducting surface, this external charge will influence the charges of the conductor and will generate on the surface of the conducting structure induced charges of opposite sign [5]. If a conductor is an infinite plane, then the distribution of surface induced charge density σ is [5]:

$$\sigma(x,y) = -\frac{q}{2\pi}\frac{d}{\left(x^2+y^2+d^2\right)^{\frac{3}{2}}}, \qquad (1)$$

where d is the minimal distance of external charge q from a conducting structure, located on the plane (x, y).

Analytical solution of the problem of induced charges, for example, for conducting structures of arbitrary geometry, is very difficult, calling for the use of numerical methods. The main feature of the latter methods is a transition from continuous variation of argu-

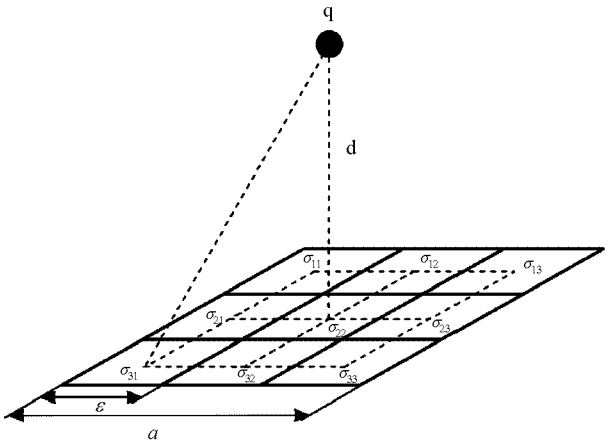

Fig. 1. Partition of conducting structure on cells

ments of the functions under consideration to their discrete values. Most frequently a net method is used for these purposes, i.e. the structure under study is divided into cells, whose coordinates are simply calculated. Net partition may be uniform or non-uniform. Non-uniform partition is used near complicated boundaries of objects. In our case we used uniform partition.

In order to solve numerically the problem of distribution of ICD on objects of finite size we divided the conducting structure, square in this case, into cells with area ε^2 (Fig.1.) The cell side is $\varepsilon = a/n$, where a is the square side, n - number of partitions. A surface-induced charge density σ_{ij} is the same in each cell and varies at transition from one cell to another. The indices i and j run over values from 1to n. The coordinates of a cell are chosen at its center. Cells with coordinates $\left((2i-1)\cdot\varepsilon/2,(2j-1)\cdot\varepsilon/2\right)$ are denoted as (i,j).

Let a conducting square in the plane (x, y) be in the field of negative charge q. Let ICD in each cell of the square (k, l) be equal to σ_{ij}. Then each cell of the conducting square is an induced charge of value $\sigma_{ij}\,\varepsilon^2$. These charges and the external charge q create in each cell equal potential for all cells of the square. It is connected with the fact that any conducting structure is an equipotential surface. For grounded conducting surfaces the potential equals zero in each cell.

In view of the aforesaid, we may write down n^2 equations for density of surface charges σ_{ij}:

$$\varphi_{ij} + \sum_{k,l} \frac{\sigma_{kl}\varepsilon^2}{\left|\vec{r}_{ij} - \vec{r}_{kl}\right|} - \frac{1}{\left|\vec{r}_{ij} + \vec{d}\right|} = 0 \qquad (2)$$

where $\vec{r}_{ij}$, $\vec{r}_{kl}$ are the vectors determining the centers of cells (i, j) and (k, l). The vector $\vec{d} = (0, 0, d)$ determines the coordinates of an external charge. The point of origin of the reference system is chosen in the center of the conducting structure. A potential induced in the center of cell by charges of the same cell is φ_{ij}. The second item in the equation (2) contributes to the potential of the cell (i, j) from all other cells of the conducting square. Summation is made over all cells, excluding the cell (i, j). The last item in the equation (2) determines the contribution to the potential of the cell (i, j) from the external charge q. It is to be mentioned that the equation (2) is written down in dimensionless units. The metrical variables are measured in the units of characteristic dimension of a conducting structure a, a charge is measured in units of external charge q, surface induced charges density is measured in units of q/a^2, and potential – in units of q/a.

In order to calculate the value φ_{ij} we use the fact that the charge density of surface charges in the selected cell is constant and equals σ_{ij}. Thus,

$$\varphi_{ij} = \sigma_{ij}\int_0^{\varepsilon}\int_0^{\varepsilon} \frac{dxdy}{\sqrt{x_{ij}^2 + y_{ij}^2}} = 3.5255\,\sigma_{ij}\,\varepsilon, \qquad (3)$$

where the coordinates x_{ij} and y_{ij} run over values from 0 to ε in the selected cell (i, j).

It should be noted that such calculations may be conducted because a potential of simple layer (6) is definite and continuous everywhere. For points which belong to the conducting surface, though the integral (6) is improper, it is absolutely convergent in this case [6].

Solution of the set of equations (2) determines σ_{ij} in each cell of a conducting structure. The 3D- chart of distribution of ICD on the surface of the conducting layer,

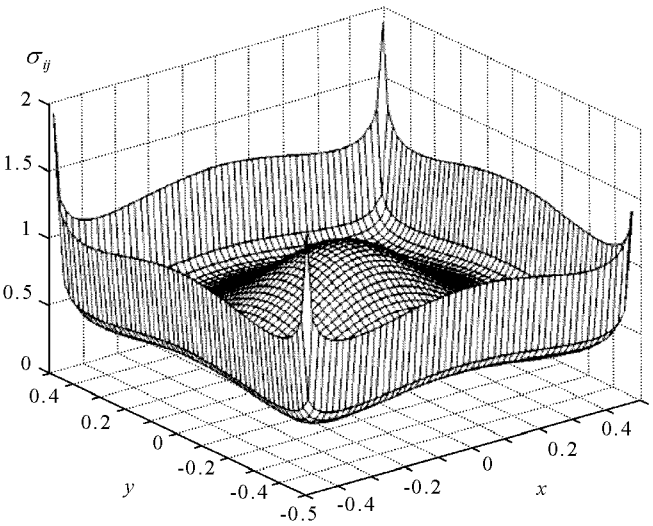

Fig. 2. Distribution of induced charge density on square

being in the field of negative single charge, is shown in Fig. 2.

Analysis of the obtained results argues that if a negative external charge is located far enough from the conducting square ($d>0.5$), then a sufficiently high density of induced positive charges is concentrated in the corners of the square, as compared to the center of the square. When the external charge comes nearer to the conducting plate, e.g. at the distance d=0.1, Coulomb fields redistribute in such a way that this plate becomes an equipotential surface. At that, big charges are induced in the center of the plate, as compared to the induced charges in its corners.

We verified the algorithm of the calculation of σ_{ij} in two ways. First, we used for verification Green's reciprocity theorem [5], which states that if conductors with charges on them $Q_1,Q_2,...Q_n$ have potentials $V_1,V_2,...V_n$, and with charges $Q_1',Q_2',...Q_n'$ have potentials $V_1',V_2',...V_n'$, respectively , then the following relation is valid:

$$\sum_{i=1}^{n} Q_i V_i' = \sum_{i=1}^{n} Q_i' V_i \qquad (4)$$

We applied the relation (4) to the system "grounded conductor – external charge". The value of induced charge on the grounded conductor in this case equals:

$$Q = -\frac{V_q'}{V'} q \qquad (5)$$

where V_q' is the value of potential at the point of location of external charge in the absence of external charge q, and the potential of the conducting structure is V'.

The second verification consists in the implementation in some approximation of the case of infinite plane at calculation of induced charge density. We moved the external charge q maximally nearer to the conducting square. As a result, the induced charge density on the plate was found to vary with distance from external charge according to the law (1), as was expected in the case of infinite plane.

Both verifications confirmed the accuracy of the elaborated algorithm of calculation of surface induced charge density. The obtained algorithm may be applied to conductors of arbitrary geometry being under potential. External charge may be extended to a conducting structure of arbitrary geometry.

To calculate ICD on objects of finite size with holes of arbitrary form we use the fact that an induced charge in holes equals zero, i.e. in respective cells of net partition σ_{ij}=0. Here we consider conducting simply connected structures, being equipotential surfaces. The number of equations of the set (2) decrease just by the number of cells in holes. The number of remaining equations exactly equals the number of unknown variables σ_{ij} in conducting cells (i, j). Thus, we obtain a unique solution of the set of equations (2).

When we build an algorithm of calculation of induced charge density on conducting structures of arbitrary geometry we introduce into the consideration a correspondence matrix with elements 1 and 0. A set of elements of this matrix a_{ij} is in biunique correspondence with the set of cells of net partition. A non-zero ICD $\sigma_{ij} \neq 0$ for the cell (i, j) corresponds to elements for which $a_{ij} = 1$ and σ_{ij}=0 corresponds to elements $a_{kl} = 0$. In such a way a structure of arbitrary geometry may be specified, for which induced charge in holes equals zero.

Correspondence matrix is used to construct a set of linear algebraic equations (SLAE). For each cell of net partition with $a_{ij} = 1$ an equation for σ_{ij} is derived. Charge densities of cells, for which $a_{kl} = 0$ do not appear in the SLAE. As a result we obtain a set of equations (2). The given SLAE may be solved by any known method of solution of SLAE [7]. We used the triangulation method in calculations. Therefore, solution of SLAE is a value set σ_{ij} for cells, where $a_{ij} = 1$.

Results of numerical calculations

For example, we calculate induced charge density on a conducting structure of Sierpinski carpet type [8], being under the influence of external point charge. This structure has been selected by us because at present fractal electrodynamic structures are widely used in modern radiophysical systems, for example as elements of fractal electronics (fractal antennas, fractal condensers, fractal resistors etc). Analytical solution of the problem is very difficult, therefore we use the above described numerical algorithm for its solution.

The results of calculation of distribution of ICD on the fractal structure of Sierpinski carpet type are given in Fig. 3. The cells of equal color correspond to equal value density of induced charge; darker color corresponds to lesser induced charge density. The calculations were carried out at the distance of external charge d= 0.4. Induced charge density well reflects the structure of the 3rd iteration of the Sierpinski carpet. Dark squares are well seen, for which σ_{ij}=0 and respective elements of correspondence matrix also equal zero $a_{kl} = 0$.

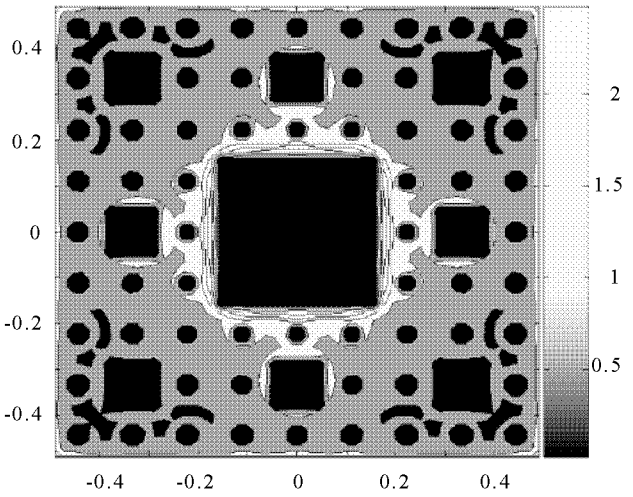

Fig. 3. Distribution of induced charge density on a fractal Sierpinski carpet type

The elaborated algorithm allows calculation of the dependence of distribution of ICD at distance d from an external charge. The calculations were performed for diagonal cells of a conducting square without holes. The respective results are shown in Fig. 4. The calculations show that distribution of charge density drastically changes on the diagonal of the square. When a negative charge moves nearer to the plate, the induced positive charge, which was concentrated at its corners, shifts to the center of the conducting square.

A total induced charge is a very important characteristic of a conducting electrodynamic structure. If we know the induced charge density, we may easily calculate the total induced charge Q on conducting plates of various geometries depending on the distance d of external charge from these structures.

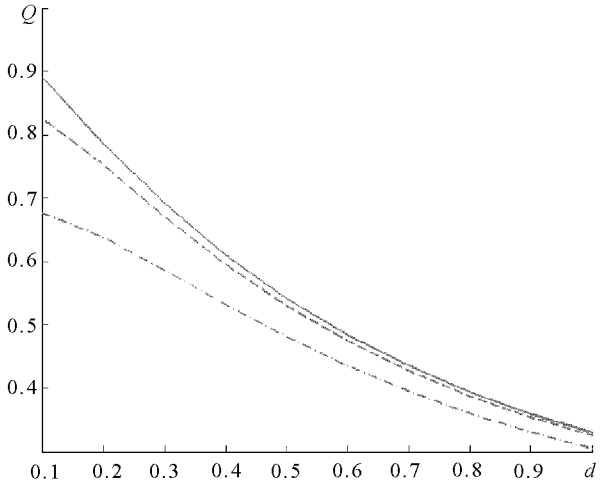

Fig. 5. Dependence of total induced charge Q on distance d from external unit charge: square (solid line); Sierpinski carpet, 3rd iteration (dashed line); square diaphragm (dash-dotted line).

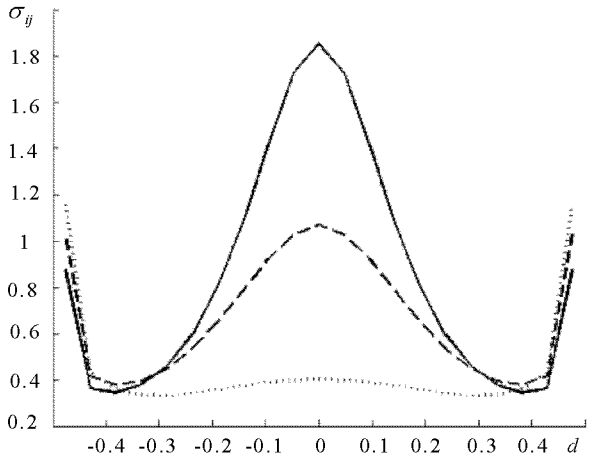

Fig. 4. Distribution of surface induced charge density on a square diagonal for various distances d from external charge d=0.3 (solid line); d=0.4 (dashed line); d=0.7 (dash-dotted line).

In order to determine the total induced charge we must sum the induced charges over all cells, constituting the conducting structure, for which $a_{ij} = 1$. The results of calculation are given in Fig. 5. They show that the total induced charge increases when external charge moves nearer to conducting structures of various geometries. We have considered three conducting structures: square, 3rd iteration of Sierpinski carpet and square diaphragm, i.e. square with square hole in its center. The results show that when an external charge moves near the plate of arbitrary geometry the total induced charge increases. Its increase is related to the area of the conducting surface of the electrodynamic structure. Values Q for square diaphragm and for Sierpinski carpet with equal d are less than for a solid square. This may be explained by the lesser effective area of these structures, therefore lesser quantity of conduction electrons comes under the influence of external charge. Calculations on square diaphragms have shown that with an increase of the size of hole in it the total induced charge decreases respectively for all d.

In the case when an external charge is at distance $d << 1$ the case approximate infinite plane is realized. In fact, the calculations have shown that in the case the total induced charge Q=1, i.e. equals by module to external charge, as was to be expected. The results confirm the accuracy of the algorithm of calculation of ICD.

If we know the distribution of induced surface charge density, we may easily calculate the spatial distribution of the induced potential φ, according to the formula:

$$\varphi = \sum_{i,k} \frac{\sigma_{ik} \varepsilon^2}{\left| \vec{R} - \vec{r}_{ij} \right|}, \qquad (6)$$

where $\vec{R}$ is the vector of point of space in which the induced potential is calculated. Summation is performed

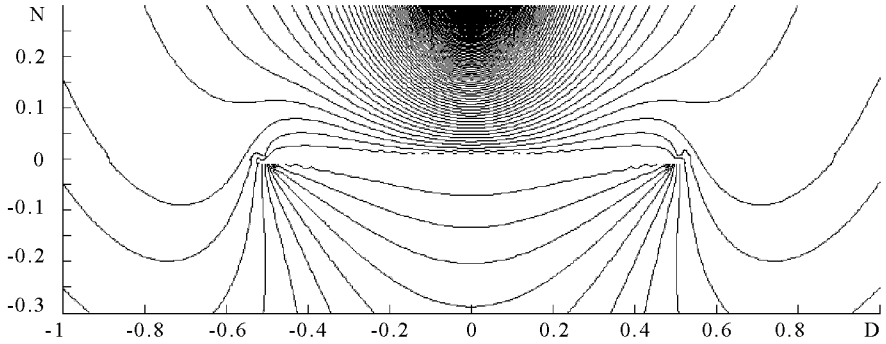

Fig. 6. Equipotential curves in the plane transiting over square diagonal

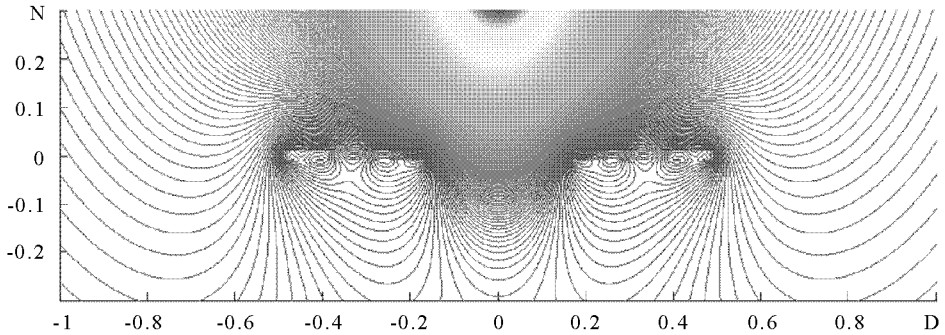

Fig. 7. Equipotential curves in the plane transiting over the diagonal of a fractal of Sierpinski carpet type

over all cells, where $\sigma_{ij} \neq 0$. As earlier, the point of origin of coordinates is located in the center of the conducting structure.

For calculation of equipotential surfaces we used the formula:

$$\varphi + \varphi_q = const, \qquad (7)$$

where φ_q is a potential in the point $\vec{R}$, generated by external inducing charge. The sections of equipotential surfaces illustrate this (Fig. 6 and Fig. 7).

Equipotential curves, located in the plane transiting over the diagonal of a conducting square and perpendicular to the plane on which it is located, are given in Fig. 6.

Equipotential curves, located in the plane transiting over the diagonal of a fractal of Sierpinski carpet type and perpendicular to the plane on which it is located, are given in Fig. 7.

The equipotential curves, given in Fig. 6 and Fig. 7 well determine the location of external charge (z=0.3) and conducting confined structure (z=0). Knowledge of the spatial distribution of induced potentials φ of a conducting structure permits calculation of induced electric fields $\vec{E}$ of any point of space by the well-known formula: $\vec{E} = -grad\varphi$.

The performed calculations mainly refer to solid conducting square, square diaphragm and 3rd iteration of Sierpinski carpet. The algorithm proposed in the paper may be used for calculation of ICD for any plane figure of arbitrary geometry. Currently we are elaborating a program for calculation of induced charge density for confined surfaces of arbitrary non-plane geometry.

Conclusion

The paper proposes a method of numerical solution of the problem of electrostatics – calculation of the distribution of induced charge density on planar conducting structures of arbitrary geometry. If we know the value and distribution of induced charge density, we can calculate the spatial distribution of induced potentials and electric fields. The elaborated algorithm may also be applied to calculation of the distribution of charges on any conducting surface, being under a given potential.

The algorithm also enables to solve more complicated electrostatic problems, e.g. problems of determination of spatial distribution of electric fields of many confined conducting bodies of various geometries and being under various potentials.

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შემოთავაზებულია ინდუცირებული მუხტების სიმკვრივის განაწილების გამოთვლის მეთოდი ზედაპირებისათვის, მათ შორის სერპინსკის ხალიჩის მსგავსი ფრაქტალური ზედაპირებისათვის. გამოთვლის სქემა ეფუძნება იმ ფაქტს, რომ ნებისმიერი გეომეტრიის მარტივი შეერთებების მქონე გამტარი ზედაპირი წარმოადგენს ეკვიპოტენციალურ ზედაპირს.

REFERENCES

1. *J. Ouyang, C.-W. Chu, D. Sieves, Y. Yang*. (2005), Appl. Phys. Letter. **86**: 123507.
2. *E. T. Yu et al*. (1997), Appl. Phys. Lett.. **71,** 19: 2794–2796.
3. *M. Z. Bazant, T. M. Squires*. (2004), Phys. Rev. letter. **9**, 6: 066101.
4. *Y. Tjmita, R. Smirnov, Z. Sizheng*. (2006), Plasma Science & Technology, **8**, 1: 122–124.
5. *V. Smait*. (1954), Elektrostatika i elektrodinamika, M., 606 p (in Russian).
6. *V. Ya. Arsenin*. (1974), Metody matematicheskoi fiziki i specialnye funkcii. M., 432 p. (in Russian).
7. *V. A. Ilina, P. K. Silaev*. (2003), Chislennye metody dlya fizikov-teoretikov. Moscow-Izhevsk, 132 p. (in Russian).
8. *E. Fiodorov*. (1991), Fraktaly. M., 264 p. (in Russian).