Mathematics

Integral Representations of Analytic and Pluriharmonic Functions of the Bergman Class in the Unit Ball

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ABSTRACT. We consider analytic and pluriharmonic functions of the Bergman class defined in the unit ball. The presented theorems provide representations of functions of this class. It is shown that nth order fractional integrals of these functions possess finite K-limits almost everywhere on the sphere. © 2007 Bull. Georg. Natl. Acad. Sci.

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Following the notation of Rudin [1, 2], we denote by $C$ the space of complex numbers and $C^n = \{ z = (z_1, z_2, \ldots, z_n), \ z_j \in C, \ j = 1, n \}$.

We denote by $B$ the unit ball in the space $C^n$, and by $S$ the sphere, i.e.

$$ B = \left\{ z \in C^n : \sum_{j=1}^{n} |z_j|^2 < 1 \right\}, $$

$$ S = \partial B = \left\{ z \in C^n : |z| = 1 \right\}. $$

A point $z \in C^n$ is written in the form $z = r \cdot t$, where $r = |z|$ and $t = \frac{z}{|z|}$. The normalized Lebesgue measure $d\mu(z) = c \ dx_1 dy_1 \cdots dx_n dy_n$ is written in terms of spherical coordinates $(r, t)$ as $d\mu(z) = 2nr^{2n-1} \ dr \ d\sigma(t)$, where $d\sigma(t)$ is the induced Lebesgue measure on $S$ normalized so that

$$ \int_{B} d\mu(z) = \int_{0}^{2\pi} \int_{0}^{\pi} r^{2n-1} \ dr \ d\sigma(t) = 1. $$

Denote by $RP(G)$ the class of all functions in $\Omega$ that are the real parts of analytic functions.

Definition 1. Let $0 < p < \infty$. An analytic function $f$ in $B$ is said to belong to $H^p_f(B)$ if

\[ \| f \| = \left( \int_{B} |f(z)|^p \, d\mu(z) \right)^{\frac{1}{p}} < \infty. \]

**Definition 2.** The set of pluriharmonic functions in \( B \) such that

\[ \| u \| = \left( \int_{B} |u(z)|^p \, d\mu(z) \right)^{\frac{1}{p}} < \infty \]

is called the class \( h^p(B) \) \( (0 < p < \infty) \).

**Definition 3.** Let \( D \) be an open set in \( C^n \). A function \( u \in C^2(D) \) is called pluriharmonic if it satisfies the differential equations

\[ \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} = 0, \quad j = 1, n, \quad k = 1, n. \]

Rudin, Nagel and Forelli established that a function \( u : B \to R \), where \( u \in C^2(B) \), is pluriharmonic if and only if \( u \in RP(B) \) [2].

Consider the functions

\[ K(z,t) = \left(1 - \langle z, t \rangle \right)^{n-1}, \quad z \in B, \quad t \in S; \]
\[ H(z,t) = 2K(z,t) - 1, \quad z \in B, \quad t \in S; \]
\[ P(z,t) = \text{Re } H(z,t), \quad z \in B, \quad t \in S; \]

where \( \langle z, t \rangle = \sum_{j=1}^{n} z_j \bar{t}_j, \ z = (z_1, ..., z_n), \ t = (t_1, ..., t_n). \)

Let \( f \in L^p(B), \ p \geq 1 \), and introduce the operators

\[ K[f](z) = \int_{B} K(z,t) f(t) \, d\mu(t); \]
\[ H[f](z) = \int_{B} H(z,t) f(t) \, d\mu(t); \]
\[ P[f](z) = \int_{B} P(z,t) f(t) \, d\mu(t). \]

It is obvious that \( K[f] \in H(B), \ H[f] \in H(B) \) and \( P[f] \in RP(B) \) (if \( f : B \to R \)), where \( H(B) \) is the space of analytic functions in \( B \).

For \( \alpha > 1 \) and \( t \in S \) we denote by \( D_\alpha(t) \) the set of all points \( z \in C^n \) such that

\[ |1 - \langle z, t \rangle| < \frac{\alpha}{2} \left(1 - |z|^2 \right). \]  

(1)

It clearly follows that \( \overline{D_\alpha(t)} \in B \). When \( \alpha \leq 1 \), the inequality (1) gives an empty set.

Assume that \( t \in S \), \( \Omega \) is open in \( B \) and to each number \( \alpha > 1 \) there corresponds a number \( r < 1 \) such that
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\[ \{ |z| > r \} \cap D_\alpha(t) \subset \Omega \]

(\( \Omega = B \) is the simplest and most important example).

**Definition 4.** We say that \( \lambda \) is the \( K \)-limit at the point \( t \) for the function \( F : \Omega \to C \), i.e.

\[ (K - \lim F)(t) = \lambda \]  

(2)

if for any \( \alpha > 1 \) and any sequence of points \( \{z_j\} \) from \( D_\alpha(t) \cap \Omega \) that converges to \( t \), \( F(z_j) \to \lambda \) as \( j \to \infty \).

If \( f \in H(B) \) and \( \alpha > 1 \), then the function

\[ f_{[\alpha]}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k + 1)}{\Gamma(k + \alpha + 1)} b_k(z) \]

is called a fractional integral of order \( \alpha \) in the sense of Hahn and Mitchell [3], where \( \sum_{k=0}^{\infty} b_k(z) \) is a homogeneous expansion of the functions \( f \) in \( B \), and \( \Gamma \) is the Euler function.

The following theorems are valid.

**Theorem 1.** The conditions

\[ \int_B f(t) \left( \overline{t}, z \right)^m dt = 0, \quad \forall z \in B \quad \text{and} \quad \forall m \in \mathbb{N}, \]

are necessary and sufficient for \( \mathcal{P} \left[ f \right](z) \) to be an analytic function inside the unit ball.

**Theorem 2.** Let \( f \in \mathcal{R}^p(B) \). Then the following propositions are equivalent:

1) \( f \in h_1(B) \).
2) \( \forall z \in B, \quad f(z) = \mathcal{P} \left[ f \right](z) \).

**Theorem 3.** Let \( f \in H_p^1(B), \quad p \geq 1 \). Then for \( \forall z \in B \)

\[ f(z) = H_1 \left[ f \right](z) + i \operatorname{Im} f(0), \]

where \( u = \operatorname{Re} f \).

**Theorem 4.** Let \( f \in H(B) \). Then the following propositions are equivalent:

1) \( f \in H_1(B) \).
2) \( \forall z \in B, \quad f(z) = K_1 \left[ f \right](z), \)
3) \( \forall z \in B, \quad f(z) = \mathcal{P} \left[ f \right](z) \).

**Theorem 5.** Let \( L_1(B) \) and \( C_1 \left[ f \right](z) = \int_B \frac{f(t) dt}{1 - \langle z, t \rangle} \). Then

\[ C(f) \in \bigcap_{0 < q < 1} H_q(B), \]

where \( H_q(B) \) is the Hardy space.

**Corollary 1.** If \( f \in L_p^q(B) \) (\( p \geq 1 \)) and \( F(z) = K_1 \left[ f \right](z) \), then the function \( F_{[n]}(z) \), where \( n = \dim \mathbb{C}^n \), has the finite \( K \)-limit almost everywhere on \( S \).

**Corollary 2.** If \( f \in H_p^1(B), \quad p \geq 1 \), then the function \( F_{[n]} \) has the finite \( K \)-limit almost everywhere on \( S \).

**Theorem 6.** If \( f \in h_p^1(B), \quad p \geq 1 \), then the function \( F_{[n]} \) has the finite \( K \)-limit almost everywhere on \( S \).
**Theorem 7.** If $f \in L^p(B)$ and $p > 1$, then the operator $u = P[f](z)$ is the bounded operator from the space $L^p(B)$ to $h^p(B)$.

### References


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