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Mathematics

Integral Representations of Analytic and Pluriharmonic Functions of the Bergman Class in the Unit Ball

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ABSTRACT. We consider analytic and pluriharmonic functions of the Bergman class defined in the unit ball. The presented theorems provide representations of functions of this class. It is shown that *n*th order fractional integrals of these functions possess finite *K*-limits almost everywhere on the sphere.

Key words: *Bergman class, pluriharmonic function, Hardy class, K-limit, fractional integral.*

Following the notation of Rudin [1, 2], we denote by C the space of complex numbers and $C^n = \{z : z = (z_1, z_2, \ldots, z_n),\ z_j \in C,\ j = \overline{1,n}\}$.

We denote by B the unit ball in the space C^n, and by S the sphere, i.e.

$$B = \left\{ z \in C^n : |z|^2 = \sum_{j=1}^{n} |z_j|^2 < 1 \right\},$$

$$S = \partial B = \{ z \in C^n \ \ |z| = 1 \}.$$

A point $z \in C^n$ is written in the form $z = r \cdot t$, where $r = |z|$ and $t = \frac{z}{|z|}$. The normalized Lebesgue measure $d\mu(z) = c\, dx_1\, dy_1 \cdots dx_n dy_n$ is written in terms of spherical coordinates (r,t) as $d\mu(z) = 2nr^{2n-1}\, dr\, d\sigma(t)$, where $d\sigma(t)$ is the induced Lebesgue measure on S normalized so that

$$\int_B d\mu(z) = \int_0^0 \int_S r^{2n-1}\, dr\, d\sigma(t) = 1.$$

Denote by $RP(G)$ the class of all functions in Ω that are the real parts of analytic functions.

Definition 1. Let $0 < p < \infty$. An analytic function *f* in B is said to belong to $H'_p(B)$ if

$$\| f \| = \left[\int_B | f(z) |^p \, d\mu(z) \right]^{\frac{1}{p}} < \infty .$$

Definition 2. The set of pluriharmonic functions in B such that

$$\| u \| = \left[\int_B | u(z) |^p \, d\mu(z) \right]^{\frac{1}{p}} < \infty$$

is called the class $h_p^{'}(B)$ $(0 < p < \infty)$.

Definition 3. Let D be an open set in C^n. A function $u \in C^2(D)$ is called pluriharmonic if it satisfies the differential equations

$$\frac{\partial^2 u}{\partial z_j \, \partial \bar{z}_k} = 0, \quad j = \overline{1,n}, \quad k = \overline{1,n} .$$

Rudin, Nagel and Forelli established that a functon $u : B \to R$, where $u \in C^2(B)$, is pluriharmonic if and only if $u \in RP(B)$ [2].

Consider the functions

$$K(z,t) = \left(1 - \langle z,t \rangle\right)^{-n-1}, \quad z \in B, \quad t \in S;$$
$$H(z,t) = 2K(z,t) - 1, \quad z \in B, \quad t \in S;$$
$$P(z,t) = \operatorname{Re} H(z,t), \quad z \in B, \quad t \in S;$$

where $\langle z,t \rangle = \sum_{j=1}^{n} z_j \bar{t}_j$, $z = (z_1, \ldots, z_n)$, $t = (t_1, \ldots, t_n)$.

Let $f \in L^p(B)$, $p \geq 1$, and introduce the operators

$$K[f](z) = \int_B K(z,t) f(t) \, d\mu(t);$$
$$H[f](z) = \int_B H(z,t) f(t) \, d\mu(t);$$
$$P[f](z) = \int_B P(z,t) f(t) \, d\mu(t).$$

It is obvious that $K[f] \in H(B)$, $H[f] \in H(B)$ and $P[f] \in RP(B)$ (if $f : B \to R$), where $H(B)$ is the space of analytic functions in B.

For $\alpha > 1$ and $t \in S$ we denote by $D_\alpha(t)$ the set of all points $z \in C^n$ such that

$$\left| 1 - \langle z,t \rangle \right| < \frac{\alpha}{2} \left(1 - | z |^2\right). \tag{1}$$

It clearly follows that $\overline{D}_\alpha(t) \in B$. When $\alpha \leq 1$, the inequality (1) gives an empty set.

Assume that $t \in S$, Ω is open in B and to each number $\alpha > 1$ there corresponds a number $r < 1$ such that

$$\{|z|>r\}\cap D_\alpha(t)\subset\Omega$$

($\Omega=B$ is the simplest and most important example).

Definition 4. We say that λ is the *K*-limit at the point *t* for the function $F:\Omega\to C$, i.e.

$$\left(K-\lim F\right)(t)=\lambda \tag{2}$$

if for any $\alpha>1$ and any sequence of points (z_j) from $D_\alpha(t)\cap\Omega$ that converges to *t*, $F(z_j)\to\lambda$ as $j\to\infty$.

If $f\in H(B)$ and $\alpha>1$, then the function

$$f_{[a]}(z)=\sum_{k=0}^{\infty}\frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)}b_k(z)$$

is called a fractional integral of order α in the sense of Hahn and Mitchell [3], where $\sum_{k=0}^{\infty}b_k(z)$ is a homogeneous expansion of the functions *f* in *B*, and Γ is the Euler function.

The following theorems are valid.

Theorem 1. *The conditions*

$$\int_B f(t)\left(\langle t,z\rangle\right)^m d\mu(t)=0,\quad \forall z\in B \quad and \quad \forall m\in N,$$

are necessary and sufficient for $P[f](z)$ *to be an analytic function inside the unit ball.*

Theorem 2. *Let* $f\in RP(B)$. *Then the following propositions are equivalent:*

1) $f\in h_1'(B)$,

2) $\forall z\in B,\ f(z)=P[f](z)$.

Theorem 3. *Let* $f\in H_p'(B),\ p\geq 1$. *Then for* $\forall z\in B$

$$f(z)=H[f](z)+i\,\mathrm{Im}\,f(0),$$

where $u=\mathrm{Re}\,f$.

Theorem 4. *Let* $f\in H(B)$. *Then the following propositions are equivalent:*

1) $f\in H_1'(B)$,

2) $\forall z\in B,\ f(z)=K[f](z)$,

3) $\forall z\in B,\ f(z)=P[f](z)$.

Theorem 5. *Let* $L^1(B)$ *and* $C[f](z)=\int_B\frac{f(t)\,d\mu(t)}{1-\langle z,t\rangle}$. *Then*

$$C(f)\in\bigcap_{0<q<1}H^q(B),$$

where $H^q(B)$ *is the Hardy space.*

Corollary 1. *If* $f\in L^p(B)$ $(p\geq 1)$ *and* $F(z)=K[f](z)$, *then the function* $F_{[n]}(z)$, *where* $n=\dim C^n$, *has the finite K-limit almost everywhere on S.*

Corollary 2. *If* $f\in H_p'(B),\ p\geq 1$, *then the function* $f_{[n]}$ *has the finite K-limit almost everywhere on S.*

Theorem 6. *If* $f\in h_p'(B),\ p\geq 1$, *then the function* $f_{[n]}$ *has the finite K-limit almost everywhere on S.*

Theorem 7. *If* $f \in L^p(B)$ *and* $p > 1$, *then the operator* $u = P[f](z)$ *is the bounded operator from the space* $L^p(B)$ *to* $h_p^{'}(B)$.

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