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Cybernetics

On the Method of Modeling of a Stationary Sequence

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ABSTRACT. Methods of modeling of stationary Gaussian or non-Gaussian sequences of some classes are considered. This problem is reduced to the modeling of independent stationary Gaussian Markov sequences. Simple algorithms of modeling of these sequences are presented. *© 2007 Bull. Georg. Natl. Acad. Sci.*

Key words: *stationary process, Gaussian process, Markov process, process of class*-N*, log-normal distribution, Johnson's* S_B *distribution.*

Let $\eta(t,\omega)$ be a stationary (in a wide sense) real-valued random process with mean 0 and covariation function

$$B(t,s) = B(t-s) = \sigma^2 e^{-\alpha|t-s|}\left[\cos(\beta t)\cos(\beta s) + \sin(\beta t)\sin(\beta s)\right], \quad \alpha > 0, \tag{1}$$

Obviously

$$\eta(t,\omega) = \sigma\cos(\beta t)\eta_1(t,\omega) + \sigma\sin(\beta t)\eta_2(t,\omega), \tag{2}$$

where $\eta_1(t,\omega)$ and $\eta_2(t,\omega)$ are stationary (in a wide sense) real-valued mutually uncorrelated random processes, whose autocorrelation function according to (1) and (2) will be

$$B_{\eta_k\eta_k}(t-s) = e^{-\alpha|t-s|}, \quad \alpha > 0,\ k = 1,2. \tag{3}$$

In the case when $\eta(t,\omega)$ is a stationary (in a wide sense) Gaussian real-valued process, then $\eta_1(t,\omega)$ and $\eta_2(t,\omega)$ are mutually independent stationary Gaussian real-valued Markov processes with mean 0, variance 1 and one-step correlation according to (3) [1]

$$B_{\eta_k\eta_k}(1) = e^{-\alpha} = r_1, \quad \alpha > 0,\ k = 1,2.$$

If $\eta(t,\omega)$ is a stationary Gaussian sequence, then an algorithm of simulation of the sequences $\eta_1(t,\omega)$ and $\eta_2(t,\omega)$ will be [2]

$$\eta_1^{(1)} = \zeta_1^{(1)},$$

$$\eta_{i+1}^{(1)} = r_1 \eta_i^{(1)} + \sqrt{1-r_1^2}\, \zeta_{i+1}^{(1)}, \tag{4}$$

$$\eta_1^{(2)} = \zeta_1^{(2)},$$

$$\eta_{i+1}^{(2)} = r_1 \eta_i^{(2)} + \sqrt{1-r_1^2}\, \zeta_{i+1}^{(2)}, \quad i = 1,2,...,$$

where $\zeta_i^{(1)}, \zeta_i^{(2)}$, $i = 1,2,...$ are independent sequences of independent Gaussian random variables with mean 0 and variance 1. In (4) we use the notation $\eta_1(i,\omega) = \eta_i^{(1)}$, $\eta_2(i,\omega) = \eta_i^{(2)}$.

When the covariation function has a general form

$$B(t-s) = \sigma^2 e^{-\alpha|t-s|} \sum_{k=0}^{m} A_k \cos[k(t-s)] = \sigma^2 e^{-\alpha|t-s|} \sum_{k=0}^{m} A_k [\cos(kt)\cos(ks) + \sin(kt)\sin(ks)],$$
$$A_k \ge 0, \quad k = 0,1,...,m, \tag{5}$$

then the random process $\eta(t,\omega)$ has the form :

$$\eta(t,\omega) = \sigma \sum_{k=0}^{m} \sqrt{A_k} \left[\cos(kt)\eta_k^{(1)}(t,\omega) + \sin(kt)\eta_k^{(2)}(t,\omega)\right], \tag{6}$$

where $\eta_k^{(1)}(t,\omega)$ and $\eta_k^{(2)}(t,\omega)$ are uncorrelated stationary sequences with autocorrelation function

$$B_{\eta_k^{(1)}\eta_k^{(1)}}(t-s) = e^{-\alpha|t-s|}, \ k = 0,1,2,...,m, \quad B_{\eta_k^{(2)}\eta_k^{(2)}}(t-s) = e^{-\alpha|t-s|}, \ k = 0,1,...,m,$$

$$B_{\eta_k^{(1)}\eta_i^{(1)}}(t,s) = B_{\eta_k^{(2)}\eta_i^{(2)}}(t,s) = 0, \ i \ne k, \quad B_{\eta_k^{(1)}\eta_i^{(2)}}(t,s) = 0, \quad \text{for each } i,k,t,s.$$

When $\eta(t,\omega)$ is a stationary Gaussian process, then $\{\eta_k^{(1)}(t,\omega), \eta_k^{(2)}(t,\omega), k = 0,1,...,m\}$ is a 2(m+1)-dimensional vector process of mutually independent stationary Gaussian Markov processes with mean 0, variance 1 and one-step correlation coefficient:

$$\rho_{\eta_k^{(1)}\eta_k^{(1)}}(1) = \rho_{\eta_k^{(2)}\eta_k^{(2)}}(1) = e^{-\alpha}, \ k = 0,1,...,m.$$

When $\eta(t,\omega)$ is a stationary Gaussian sequence, then simulation of each kth component of sequences $\eta_k^{(1)}(t,\omega), \eta_k^{(2)}(t,\omega), k = 0,1,...,m$ can be realized by (4).

Simulation of a random process $\xi(t,\omega)$ of class N could be defined by the formula [2,3]

$$\xi(t,\omega) = F_t^{-1}(\Phi(\eta(t,\omega))), \tag{7}$$

where $F_t^{-1}(.)$ is an inverse function of $F(x,t)$ with respect to x. $F(x,t)$ is a one-dimensional distribution function of the process $\xi(t,\omega)$ while

$$\eta(t,\omega) = \Phi^{-1}(F(\xi(t,\omega),t)) \tag{8}$$

is a Gaussian process with mean 0 and variance 1.

When one-dimensional distribution is log-normal, then [2-4]

$$F(x,t)=\begin{cases}\Phi\left(\dfrac{\ln\left(x-a(t)\right)-m(t)}{\sigma(t)}\right), & \text{when } x>a(t),\\ 0, & \text{when } x\le a(t),\end{cases} \tag{9}$$

where

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-z^2/2}\,dz.$$

In this case (7) and (8) have the forms

$$\xi(t,\omega)=\exp\{\sigma(t)\eta(t,\omega)+m(t)\}+a(t), \tag{10}$$

$$\eta(t,\omega)=\frac{\ln\left(\xi(t,\omega)-a(t)\right)-m(t)}{\sigma(t)}. \tag{11}$$

If $F(x,t)$ is Johnson's S_B distribution, then

$$F(x,t)=\begin{cases}0, & \text{when } x\le a(t),\\ \Phi\left(\dfrac{\ln\left(\dfrac{x-a(t)}{b(t)-x}\right)-m(t)}{\sigma(t)}\right), & \text{when } a(t)<x<b(t),\\ 1, & \text{when } x\ge b(t)\end{cases} \tag{12}$$

In this case (7) and (8) have the forms:

$$\xi(t,\omega)=\frac{b(t)\exp\{\sigma(t)\eta(t,\omega)+m(t)\}+a(t)}{\exp\{\sigma(t)\eta(t,\omega)+m(t)\}+1}, \tag{13}$$

$$\eta(t,\omega)=\frac{1}{\sigma(t)}\left[\ln\left(\frac{\xi(t,\omega)-a(t)}{b(t)-\xi(t,\omega)}\right)-m(t)\right]. \tag{14}$$

Simulation of a stationary Gaussian sequence $\eta(t,\omega)$ in (7), (10) and (13) will be produced by the above-mentioned method.

The method may be successfully applied in stochastic hydrology, for mathematical modeling of average annual discharge of river flow in the form of a stationary sequence.

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