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Mathematics

The Exact Homology Groups of a Complete Distributive Lattice

Zurab Todua

A. Razmadze Mathematical Institute, Tbilisi

(Presented by Academy Member N. Berikashvili)

ABSTRACT. On a category of pairs of complete distributive lattices groups of homologies are constructed over a discrete group of coefficients, which satisfy the axiom of exactness. *© 2007 Bull. Georg. Natl. Acad. Sci.*

Key words: *lattice, homology groups, prime (co)ideal.*

Let L and L' be complete distributive lattices. If the lattice v-homomorphism is an epimorphism, then is called a pair of distributive lattices. We denote it by (L, L'). Let $M \xrightarrow{j} M'$ be another pair, then the mapping of pairs $(f, f'):(L, L') \to (M.M')$ is a pair of lattice v-homomorphisms, such that (see [1]).

Let L be a complete distributive lattice and $S(L)$ be a complete lattice of its subspaces; the element e in $S(L)$ is the pair $e = (L', f')$, where is an epimorphic lattice v-homomorphism; is the closure of e in $S(L)$; $FS(L)$ are the sublattices in $S(L)$, consisting of all closed elements (see [2, 3]).

Definition 1. The ideal P of the lattice $FS(L)$ is said to be admissible for L, if for every $F_0^{'} \in P$, $F_0^{'} \neq 0$, there exist $F \in FS(L)$ and such that , $F_0^{'} \wedge F = 0$ and $F^{'} \vee F = 1$ ($P = FS(L)$, if and only if is compact).

If P is an admissible ideal for L, is a closed subspace in L, then is the admissible ideal for L'.

Example 1. If P is a set of all compact subspaces of locally compact Hausdorff space X, then P is admissible for the lattice $L(X)$ consisting of all open subspaces X (see [4]).

Proposition 1 (Example 2). *Let L be a complete distributive lattice. Then every simple ideal P of is admissible for L.*

Definition 2. The subspace $e = (L', f')$ in L (which is the same, the element e in) is said to be bounded with respect to the admissible ideal P in L, if the closure $\bar{e}$ in $S(L)$ of the subspace e is an element in P.

Let G be a discrete Abelian group, and let E^n be a set of all possible n-member sequences , bounded with respect to the admissible ideal P of subspaces in L.

Definition 3. For every integer $q \geq 0$ a q-dimensional chain of the complete distributive lattice L with respect to the admissible ideal P over the discrete group G of coefficients is called the function $c_q : E^{q+1} \to G$ which satisfies the following conditions: (1) c_q is a skew-symmetric function, i.e., $c_q(e_0,\ldots,e_q) = 0$, when among the arguments $e_0,\ldots,e_q$ there exist at least two which coincide; $c_q(e_{k0},\ldots,e_{kq}) = \varepsilon_k c_q(e_0,\ldots,e_q)$ for any permutation k of a set of numbers $0,\ldots,q$, where ε_i denotes the sign of that permutation; (2) If $e_k = e_k^{'} \vee e_k^{''}$, where $e_k^{'} \wedge e_k^{''} = 0$, then $c_q(e_0,\ldots,e_k,\ldots,e_q) = c_q(e_0,\ldots,e_k^{'},\ldots,e_q) + c_q(e_0,\ldots,e_k^{''},\ldots,e_q)$ (additivity); (3) $c_q(e_0,\ldots,e_q) = 0$, if $\overline{e}_0 \wedge \cdots \wedge \overline{e}_q = 0$.

In the set $C_q(L,P;G)$ of all q-dimensional chains of the complete distributive lattice L with respect to the admissible ideal P over the discrete group G of coefficients we introduce the algebraic structure as follows. Let $c_q^{'}$ and $c_q^{''}$ belong to the set $C_q(L,P;G)$. The sum of two q-dimensional chains $c_q^{'}$ and $c_q^{''}$ is called the function $c_q^{'} + c_q^{''}$ defined by the equality $(c_q^{'} + c_q^{''})(e_0,\ldots,e_q) = c_q^{'}(e_0,\ldots,e_q) + c_q^{''}(e_0,\ldots,e_q)$.

It can be easily verified that the function $c_q^{'} + c_q^{''}$ satisfies the conditions (1)–(3) of Definition 3 and this addition defines an Abelian group structure on $C_q(L,P;G)$.

Proposition 2. *Let P be the admissible ideal for L. Then for every* $F' \in P$, *, there exists an open subspace U in L, such that* $F' \leq U$ *and* $\overline{U} \in P$.

Definition 4. The boundary ∂ of a chain is defined as the $(q-1)$-dimensional chain given by the equality $(\partial c_q)(e_0,\ldots,e_{q-1}) = c_q(U,e_0,\ldots,e_{q-1})$, where U is an open, bounded with respect to the admissible ideal P, subspace in L for which $\bigvee_{k=0}^{q-1} \overline{e}_k < U$.

The subspace U, satisfying the conditions of Definition 4, always exists by Proposition 2. We can show that this definition does not depend on the choice of U. The operator ∂ is the homomorphism of the group of q-chains into the group of $(q-1)$-chains $C_{q-1}(L,P;G)$.

Proposition 3. $\partial \circ \partial = 0$.

Thus we obtain the chain complex $C_*(L,P;G)$ of Abelian groups $C_q(L,P;G)$ and homomorphisms ∂.

Definition 5. The groups of homology of the complex $C_*(L,P;G)$ are called q-dimensional groups of homology of the complete distributive lattice L with respect to the admissible ideal P over the discrete group G of coefficients.

The triple (by definition) (L,L',P_L) consists of both pairs of lattices $L \xrightarrow{i} L'$, where (L',i) is a closed subspace in L, and P_L-admissible ideal of L. The v-homomorphism $f : L \to M$ defines the v-homomorphism $Ff : FS(L) \to FS(M)$ and .

Definition 6. The lattice v-homomorphism of pairs $(f,f') : (L,L') \to (M,M')$ is said to be admissible for the triples and (M,M',P_M), if $(Ff)(P_L)\subset P_M$.

Let $f:L\to M$ be an admissible mapping. For every q we define the function $f_{\#}:C_q(M,P_M;G)\to C_q(L,P_L;G)$ which is induced by the mapping f as follows:

$$(f_{\#}c_q)(e_0,\ldots,e_q)=c_q(\bar{f}(e_0),\ldots,\bar{f}(e_q)),$$

where $e_0,\ldots,e_q$ are the bounded with respect to P_L subsets in L. It can be verified that the function $f_{\#}c_q$ belongs to the group $C_q(L,P_L;G)$; $f_{\#}$ is the homomorphism of the groups and that homomorphism commutes with the operator ∂.

In particular, if is the admissible triple, then the homomorphism $i_{\#}:C_q(L',P_{L'};G)\to C_q(L,P_L;G)$ (here $F=(L',i)$ is the closed subspace in L, , and thus is the admissible ideal for L') which is, as it can be easily seen, a homomorphism. We identify the group with its image for the monomorphism $i_{\#}$. The factor-group $C_q(L,P_L;G)/C_q(L',P_{L'};G)$ is called the q-dimensional group $C_q(L,L',P_L;G)$ of chains of the pair (L,L') with respect to the admissible ideal P_L over the discrete group G of coefficients. The boundary operator is called the mapping of the group $C_q(L,L',P_L;G)$ into the group $C_{q-1}(L,L',P_L;G)$ which is induced by the homomorphism ∂. This definition implies directly that .

Thus we have obtained the chain complex of the groups $C_q(L,L',P_L;G)$ and homomorphisms ∂.

Definition 7. The group of homology of the chain complex $C_*(L,L',P_L;G)$ is called the q-dimensional group of the pair (L,L') with respect to the admissible ideal P_L over the discrete group G of coefficients.

Let the mapping $(f,f'):(L,L')\to(M,M')$ be admissible for the triples and (M,M',P_M). We define the mapping $f_{\#}:C_q(M,M',P_M;G)\to C_q(L,L',P_L;G)$ as the chain mapping induced by the mapping $(f/M)_{\#}:C_*(M,P_M;G)\to C_*(L,P_L;G)$.

Definition 8. The homomorphism induced by the admissible mapping $(f,f'):(L,L')\to(M,M')$ is called the homomorphism which is induced by the mapping $f_{\#}$.

A short exact sequence of chain complexes

$$0\to C_*(L',P_{L'};G)\to C_*(L,P_L;G)\to C_*(L,L',P_L;G)\to 0$$

defines (see [5]) the connecting homomorphism $\partial_{\#}:H_q(L,L',P_L;G)\to H_{q-1}(L',P_{L'};G)$, and we have the following

Theorem 1. *For every triple* (L,L',P_L) *the sequence of groups*

$$\cdots\to H_{q+1}(L,L',P_L;G)\to H_q(L',P_{L'};G)\to H_q(L,P_L;G)\to H_q(L,L',P_L;G)\to\cdots$$

is the exact sequence.

Particular Case 1. Let L be the lattice of all open subspaces of the locally compact Hausdorff space X; L' be the lattice of the closed subspace in X; be the ideal of all compact subsets in X. Then we obtain the exact theory of Kolmogorov homologies (see [6, 7]).

Particular Case 2. Let L be the lattice of all open subsets of an arbitrary topological space X; L' is the lattice of the closed subspace in X;   an arbitrary simple ideal in the lattice of all closed subspaces in X. Then we obtain the exact homology theory of X with respect to the simple ideal P_L .

მათემატიკა

სრული დისტრიბუციული მესერების ზუსტი ჰომოლოგიის ჯგუფები

ზ. თოდუა

ა. რაზმაძის მათემატიკის ინსტიტუტი, თბილისი

(წარმოდგენილია აკადემიის წევრის ნ. ბერიკაშვილის მიერ)

P_L

სრული დისტრიბუციული მესერების წყვილების კატეგორიაზე აგებულია ჰომოლოგიის ჯგუფები კოეფიციენტებით დისკრეტულ ჯგუფებში, რომლებიც აკმაყოფილებს სიზუსტის აქსიომას. ეს განმარტება, როცა L არის X ლოკალურად კომპაქტური ჰაუსდორფის სივრცის ყველა ღია ქვესიმრავლის მესერი, L′ — X-ის ჩაკეტილი ქვესივრცის ღია სიმრავლეების მესერი, P_1 — იდეალი, წარმოქმნილი X-ის ყველა კომპაქტური ქვესიმრავლეებისაგან, მაშინ მივიღებთ X-ის კოლმოგოროვის ზუსტ თეორიას. თუ L არის X ტოპოლოგიური სივრცის ყველა ღია ქვესიმრავლის მესერი, მიღებული ზუსტი ჰომოლოგიის თეორია ტოპოლოგიურ სივრცეთა კატეგორიაზე არის ახალი ზუსტი ჰომოლოგიის თეორია.

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