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Mathematics

Regular Conditional Probabilities and Disintegrations

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ABSTRACT. We clarify the connection between: (1) regular conditional probabilities relative to a sigma-algebra, (2) regular conditional probabilities relative to a mapping and (3) disintegrations with respect to a mapping. *© 2007 Bull. Georg. Natl. Acad. Sci.*

Key words: *regular conditional probability, disintegration.*

1. Introduction

Let $(X, \mathcal{A}, \mu)$ be a probability space and $\mathcal{A}_0, \mathcal{A}_1 \subset \mathcal{A}$ be σ-algebras. A *regular conditional probability on* $\mathcal{A}_0$ *associated with* μ *relative to* $\mathcal{A}_1$ is by definition a mapping $p : \mathcal{A}_0 \times X \to [0,1]$, which has the properties:

(RCP1) for a fixed $x \in X$ the set function $p(\cdot, x)$ is a probability measure on $\mathcal{A}_0$,

(RCP2) for a fixed $A \in \mathcal{A}_0$ the function $p(A, \cdot)$ is measurable with respect to $\mathcal{A}_1$,

(RCP3) $\mu(A \cap A_1) = \int_{A_1} p(A, x) d\mu(x), \quad \forall A \in \mathcal{A}_0, \ \forall A_1 \in \mathcal{A}_1.$

This notion was presented implicitly in [1, Theorem 3.1], as well as in [2, (iii), p. 389] (cf. also [3], [4] and [5, p. 624]). In [1,2] it was claimed that a regular conditional probability on $\mathcal{A}$ associated with μ relative to $\mathcal{A}_1$ exists whenever $\mathcal{A}_1$ is countably generated. However, in [3] it was shown that there exist countably generated σ-algebras $\mathcal{A}$ and $\mathcal{A}_1$, such that a regular conditional probability on $\mathcal{A}$ associated with μ relative to $\mathcal{A}_1$ does not exist. Explicitly, the concept of a regular conditional probability named *the conditional probability distribution relative to* $\mathcal{A}_1$ (in our notation), was introduced in [5, Ch.I, §9 (p. 27)]. It seems that the term "regular conditional probability" appeared in [6], where the first general existence result was obtained.

We deal also with the notion of a regular conditional probability relative to a measurable mapping, which is closely related with the concept of a regular conditional probability relative to a σ-algebra. Consider a measurable space $(Y, \mathcal{F})$ and a measurable mapping $\eta : X \to Y$ with the distribution $\mu_\eta := \mu \circ \eta^{-1}$; write $\mathcal{A}_\eta = \{\eta^{-1}(B) : B \in \mathcal{F}\}$.

A *regular conditional probability on* $\mathcal{A}_0$ *associated with* μ *relative to the mapping* η is by definition a mapping $q : \mathcal{A}_0 \times Y \to [0,1]$ which has the properties:

(MRCP1) for a fixed $y \in Y$ the set function $q(\cdot, y)$ is a probability measure on $\mathcal{A}_0$,

(MRCP2) for a fixed $A \in \mathcal{A}_0$ the function $q(A, \cdot)$ *is measurable* with respect to $\mathcal{F}$, (MRCP3) $\mu(A \cap \eta^{-1}(B)) = \int_B q(A,y)d\mu_\eta(y), \quad \forall A \in \mathcal{A}_0, \ \forall B \in \mathcal{F}.$

The notions of regular conditional probability relative to a σ-algebra and relative to a measurable mapping are compared in Proposition 3.5. We discuss also some results about uniqueness of regular conditional probabilities.

A *disintegration* of μ *on* $\mathcal{A}$ *with respect to* η is by definition a mapping $q:\mathcal{A}\times Y \to [0,1]$ which has the properties:

(Dis 1) for a fixed $y \in Y$ the set function $q(\cdot, y)$ is a probability measure on $\mathcal{A}$,

(Dis 2) for a fixed $A \in \mathcal{A}$ the function $q(A,\cdot)$ is measurable with respect to $\mathcal{F}$,

(Dis 3) there exists $N \in \mathcal{F}$ with $\mu_\eta(N) = 0$ such that for all $y \in Y \setminus N$ we have $\{y\} \in \mathcal{F}$ and for each fixed $y \in Y \setminus N$ the probability measure $q(\cdot, y)$ is concentrated on the 'fibre' $\eta^{-1}(\{y\})$ (*i.e.* $q(X \setminus \eta^{-1}(\{y\}), y) = 0$),

(Dis 4) $\mu(A) = \int_Y q(A,y)d\mu_\eta(y), \quad \forall A \in \mathcal{A}.$

We note that for this concept, instead of the name "disintegration" the names *regular conditional probability distribution given* η [7, p. 146] or *conditional measure* [8, p. 158] are also used.

Usually, disintegration is defined and studied in the context of topological spaces [9]. We refer the interested reader to the works [10-16, 19] related with disintegration. In [17] disintegrations of Gaussian measures on Banach spaces with respect to continuous linear mappings are investigated.

We show that whenever disintegration q on $\mathcal{A}$ with respect to η exists, a regular conditional probability on $\mathcal{A}$ associated with μ relative to η exists as well and it coincides with q (see Proposition 4.2, in which the case of validity of the converse statement is also included).

2. Measurable mappings and transition probabilities.

Let X be a set, $(Y, \mathcal{F})$ be a measurable space; for a mapping $\eta : X \to Y$ we write $\mathcal{A}_\eta := \{\eta^{-1}(F) : F \in \mathcal{F}\}$. If $\mathcal{A}$ is a σ-algebra of subsets of X, then the mapping η will be called $(\mathcal{A}, \mathcal{F})$*-measurable,* if $\mathcal{A}_\eta \subset \mathcal{A}$.

For a topological space Z we denote $\mathcal{B}(Z)$ the Borel σ-algebra of Z.

If $(X, \mathcal{A})$ is a measurable space and Z is a topological space, then a mapping $\eta : X \to Z$ will be called $\mathcal{A}$-*measurable*, if it is $(\mathcal{A}, \mathcal{B}(Z))$-measurable.

If X and Z are topological spaces, then a mapping $\eta : X \to Z$ will be called *Borel measurable*, if it is $(\mathcal{B}(X), \mathcal{B}(Z))$-measurable.

A topological space Z is called *Polish*, if it is homeomorphic to a complete separable metric space.

We need the following known statement a proof of which can be seen in [18, p. 98] and [19, p. 31].

Lemma 2.1. *Let X be a set, $(Y, \mathcal{F})$ be a measurable space and $\eta : X \to Y$ be a mapping. If Z is a Polish space and $f : X \to Z$ is a $\mathcal{A}_\eta$-measurable mapping, then there exists a $\mathcal{F}$-measurable mapping $g : Y \to Z$ such that $f = g \circ \eta$.*

Remark 2.2. In case of $(Y, \mathcal{F}) = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ and $Z = \mathbb{R}$ this lemma coincides with [5, Supplement, Theorem 1.5, p. 603]. The case of a general $(Y, \mathcal{F})$ and $Z = \mathbb{R}$ is contained in [20, Lemma 1.1.5, p. 14]. In [19, p. 30] Lemma 2.1 is called *Doob-Dynkin lemma.* An extensive discussion of the related results is contained in [21, Ch.3, §11, pp. 144-151].

Let $(X, \mathcal{A})$ be a measurable space, μ be a positive measure on $\mathcal{A}$, $(Y, \mathcal{F})$ be a measurable space and $\eta : X \to Y$ be a $(\mathcal{A}, \mathcal{F})$-measurable mapping. It is easy to see that the set function μ_η defined on $\mathcal{F}$ by the equality $\mu_\eta(B) = \mu(\eta^{-1}(B)), \ \forall B \in \mathcal{F}$ is a positive measure. The measure μ_η is called *the image of* μ *with respect to* η.

We will frequently use the next known statement (see, e.g. [22, §39, Theorem C]).

Lemma 2.3. *Let $(X, \mathcal{A}, \mu)$ be a measure space, $(Y, \mathcal{F})$ a measurable space, $\eta : X \to Y$ be a $(\mathcal{A}, \mathcal{F})$-measurable mapping, μ_η be the image of μ with respect to η and $g : Y \to \mathbb{C}$ be a $\mathcal{F}$-measurable function. Then:*

(1) $g \in L_1(Y, \mathcal{F}, \mu_\eta) \Leftrightarrow g \circ \eta \in L_1(X, \mathcal{A}, \mu)$.

(2) *If* $g \in L_1(Y, \mathcal{F}, \mu_\eta)$, *then the following change of variable formula holds:*

$$\int_{\eta^{-1}(B)} g(\eta(x))d\mu(x) = \int_B g(y)d\mu_\eta(y), \quad \forall B \in \mathcal{F}.$$

Let $(X, \mathcal{A})$ and $(Y, \mathcal{F})$ be measurable spaces. A mapping $q: \mathcal{A} \times Y \to [0,1]$ will be called *a transition probability relative to* $(X, \mathcal{A})$ *and* $(Y, \mathcal{F})$ if it has the following properties:

(TP1) for a fixed $y \in Y$ the set function $q(\cdot, y)$ is a probability measure on $\mathcal{A}$,

(TP2) for a fixed $A \in \mathcal{A}$ the function $q(A, \cdot)$ is measurable with respect to $\mathcal{F}$.

For non-empty sets X, Y and a set $E \subset X \times Y$ we write $E_{\cdot,y} := \{a \in X : (a,y) \in E\}$, $y \in Y$ and $E_{x,\cdot} := \{b \in Y : (x,b) \in E\}$, $x \in X$. For measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{F})$ we denote by $\mathcal{A} \otimes \mathcal{F}$ the σ-algebra of subsets $X \times Y$ generated by $\{A \times F : A \in \mathcal{A}, F \in \mathcal{F}\}$.

Lemma 2.4. *Let* $(X, \mathcal{A})$ *and* $(Y, \mathcal{F})$ *be measurable spaces,* $E \in \mathcal{A} \otimes \mathcal{F}$ *be a set,* $\eta : X \to Y$ *be a mapping,* $Y_0 := \eta(X)$ and $q: \mathcal{A} \times Y \to [0,1]$ *be a transition probability relative to* $(X, \mathcal{A})$ *and* $(Y, \mathcal{F})$.

Suppose further that: either,

(i) η *is* $(\mathcal{A}, \mathcal{F})$*-measurable and there exists* $C \subset (Y \setminus Y_0) \times (Y \setminus Y_0)$ *such that* $\Delta_{Y_0} \cup C \in \mathcal{F} \otimes \mathcal{F}$ *where* $\Delta_{Y_0} := \{(y,y) : y \in Y_0\}$,

or

(ii) $gr(\eta) := \{(x, \eta(x)) : x \in X\} \in \mathcal{A} \otimes \mathcal{F}$.

Then {i) ⇒ (ii) [23, Theorem 3] *and the following statements are true:*

(*a*) $E_{\cdot,y} \in \mathcal{A}, \forall y \in Y$, $E_{x,\cdot} \in \mathcal{F}, \forall x \in X$ *and* $\{\eta(x)\} \in \mathcal{F}, \forall x \in X$.

(*b*) *The function* $y \mapsto q(E_{\cdot,y}, y)$ *is well-defined and* $\mathcal{F}$*-measurable.*

(*c*) *The function* $y \mapsto q(\eta^{-1}(\{y\}), y)$ *is well-defined and* $\mathcal{F}$*-measurable.*

Proof. (*a*) The first two statements are well known (see e.g. [22, §34, Theorem A]). The last statement follows from the second one (in fact, for $E = gr(\eta)$ and $x \in X$ we have $\{\eta(x)\} = E_{x,\cdot} \in \mathcal{F}$).

(*b*) By (*a*) we have $E_{\cdot,y} \in \mathcal{A}, \forall y \in Y$, hence the function $y \mapsto f_E(y) = q(E_{\cdot,y}, y)$ is well-defined. If $E = A \times F$ is a measurable rectangle, then $f_E(y) = q(A, y)1_F(y), \forall y \in Y$ and so, our function is $\mathcal{F}$-measurable. From this the measurability of f_E for an arbitrary $E \in \mathcal{A} \otimes \mathcal{F}$ follows in the classical way (see e.g. proof of Theorem B in [22, §35]).

(*c*) follows from (*b*) applied to $E = gr(\eta)$ (as $E_{\cdot,y} = \eta^{-1}(\{y\}), \forall y \in Y$). □

Remark 2.5. In connection with this lemma it is worthwhile to note:

(1) If $\Delta_Y = \{(y,y) : y \in Y\} \in \mathcal{F} \otimes \mathcal{F}$, then for any $(\mathcal{A}, \mathcal{F})$-measurable $\eta : X \to Y$ we have $gr(\eta) \in \mathcal{A} \otimes \mathcal{F}$ (this follows from implication (i) ⇒ (ii) of Lemma 2.4).

(2) If X, Y are Polish spaces and $\eta : X \to Y$ is a mapping with $gr(\eta) \in \mathcal{B}(X) \otimes \mathcal{B}(Y)$, then η is Borel measurable [24]. More general 'topology free' versions of this statement are known too (see, e.g. [21, Theorem II.4.1, p.81]).

(3) It is not clear whether Lemma 2.4 (*c*) remains true, if merely η is $(\mathcal{A}, \mathcal{F})$-measurable and $\{\eta(x)\} \in \mathcal{F}, \forall x \in X$.

3. Regular conditional probabilities.

Let $(X, \mathcal{A}, \mu)$ be a probability space and let $\mathcal{A}_0, \mathcal{A}_1 \subset \mathcal{A}$ be σ-algebras. Consider a measurable space $(Y, \mathcal{F})$ and a $(\mathcal{A}, \mathcal{F})$-measurable mapping $\eta : X \to Y$. It is clear that a regular conditional probability q on $\mathcal{A}_0$ associated with μ relative to η is a transition probability $q: \mathcal{A}_0 \times Y \to [0,1]$ relative to $(X, \mathcal{A}_0)$ and $(X, \mathcal{F})$ that has the additional property (MRCP3).

Proposition 3.1. *Let* $(X, \mathcal{A}, \mu)$ be a *probability space,* $(Y, \mathcal{F})$ *be a measurable space,* $\eta : X \to Y$ *be a* $(\mathcal{A}, \mathcal{F})$-

measurable mapping, $q:\mathcal{A}\times Y\to[0,1]$ *be a regular conditional probability on* $\mathcal{A}$ *associated with* μ *relative to* η *and* $E\in\mathcal{A}\otimes\mathcal{F}$.

Then the mapping $x\mapsto(x,\eta(x))$ *is* $(\mathcal{A},\mathcal{A}\otimes\mathcal{F})$ *-measurable, the function* $y\mapsto q(E_{\cdot,y},y)$ *is well-defined and* $\mathcal{F}$*-measurable and the following equality is true.*

(MRCP4) $\mu\{x\in X:(x,\eta(x))\in E\}=\int_Y q(E_{\cdot,y},y)d\mu_\eta(y)$

Proof. The measurability of $x\mapsto(x,\eta(x))$ can be verified in a standard way. The function $y\mapsto q(E_{\cdot,y},y)$ is well-defined and $\mathcal{F}$-measurable by Lemma 2.4(b). If $E=A\times F$ is a measurable rectangle, then $\{x\in X:(x,\eta(x))\in E\}=A\cap\eta^{-1}(F)$ and $q(E_{\cdot,y},y)=q(A,y)1_F(y),\ \ \forall y\in Y$; so, in this case (MRCP4) coincides with (MRPC3). From this the validity of (MRCP4) for an arbitrary $E\in\mathcal{A}\otimes\mathcal{F}$ can be derived in the classical way (cf. proof of Theorem B in [22, §35]). □

We recall that a σ-algebra $\mathcal{A}$ is called *countably generated* if there exists at most countable set $\mathcal{D}\subset\mathcal{A}$ which generates $\mathcal{A}$.

Proposition 3.2. *Let* $(X,\mathcal{A},\mu)$ *be a probability space and let* $\mathcal{A}_0\subset\mathcal{A}$ *be a countably generated* σ*-algebra. Consider a measurable space* $(Y,\mathcal{F})$ *and a* $(\mathcal{A},\mathcal{F})$*-measurable mapping* $\eta:X\to Y$.

If a regular conditional probability on $\mathcal{A}_0$ *associated with* μ *relative to* η *exists, then it is essentially unique in the following natural sense:*

If q_1, q_2 *are regular conditional probabilities on* $\mathcal{A}_0$ *associated with* μ *relative to* η, *then there exists a set* $N\in\mathcal{F}$ *such that* $\mu_\eta(N)=0$ *and*

$$q_1(A,y)=q_2(A,y),\ \ \forall y\in Y\setminus N,\ \ \forall A\in\mathcal{A}_0.$$

Proof. Since the algebra generated by at most countable $\mathcal{D}$ is at most countable, we can suppose that $\mathcal{A}_0$ is generated by an at most countable algebra $\mathcal{E}=\{E_1,E_2....\}\subset\mathcal{A}_0$. Fix a natural number n. Since by (MRCP3) we have

$$\int_B q_1(E_n,y)d\mu_\eta(y)=\mu(E_n\cap\mu^{-1}(B))=\int_B q_2(E_n,y)d\mu_\eta(y)\ \ \forall B\in\mathcal{F},$$

there is $N_n\in\mathcal{F}$ such that $\mu_\eta(N_n)=0$ and $q_1(E_n,y)=q_2(E_n,y)\ \ \forall y\in Y\setminus N_n$. Let $N:=\bigcup_{n=1}^{\infty}N_n$. Then $N\in\mathcal{F}$ and $\mu_\eta(N)=0$ and $q_1(E,y)=q_2(E,y)\ \ \forall y\in Y\setminus N,\ \forall E\in\mathcal{E}$. Therefore, for a given $y\in Y\setminus N$ the probability measures $q_1(\cdot,y)$ and $q_2(\cdot,y)$ coincide on the algebra $\mathcal{E}$ generating $\mathcal{A}_0$ and consequently, they coincide on $\mathcal{A}_0$ too. □

Proposition 3.3. *Let* $(X,\mathcal{A},\mu)$ *be a probability space and let* $\mathcal{A}_0,\mathcal{A}_1\subset\mathcal{A}$ *be* σ*-algebras.*

(a) *If* $\mathcal{A}_0$ *is countably generated and a regular conditional probability on* $\mathcal{A}_0$ *associated with* μ *relative to* $\mathcal{A}_1$ *exists, then it is essentially unique in the following natural sense:*

If p_1, p_2 *are regular conditional probabilities on* $\mathcal{A}_0$ *associated with* μ *relative to* $\mathcal{A}_1$, *then there exists a set* $N\in\mathcal{A}_1$ *such that* $\mu(N)=0$ *and*

$$p_1(A,x)=p_2(A,x),\ \ \forall x\in X\setminus N,\ \ \forall A\in\mathcal{A}_0.$$

(b) If a regular conditional probability p on $\mathcal{A}$ *associated with* μ *relative to* $\mathcal{A}_1$ *exists and* $\mathcal{A}_1$ *is countably generated, then there exists a set* $N\in\mathcal{A}_1$ *such that* $\mu(N)=0$ *and*

$$p(A_1,x)=1_{A_1}(x),\ \ \forall x\in X\setminus N,\ \ \forall A_1\in\mathcal{A}_1.$$

Proof. (a) Consider the measurable space $(Y, \mathcal{F}) := (X, \mathcal{A}_1)$. Let $\eta: X \to X$ be the identity mapping. Then clearly p_1, p_2 are regular conditional probabilities on $\mathcal{A}_0$ associated with μ relative to η; hence (a) follows from Proposition 3.2.

(b) Define $p'(A_1, x) = 1_{A_1}(x)$, $A_1 \in \mathcal{A}_1$, $x \in X$. It is clear that the restriction of p to $\mathcal{A}_1 \times X$ and $p': \mathcal{A}_1 \times X \to [0,1]$ are regular conditional probabilities on $\mathcal{A}_1$ associated with μ relative to $\mathcal{A}_1$. Since $\mathcal{A}_1$ is countably generated, we can apply (a) and get the existence of a set $N \in \mathcal{A}_1$ such that $\mu(N) = 0$ and $p(A_1, x) = p'(A_1, x)$, $\forall x \in X \setminus N$, $\forall A_1 \in \mathcal{A}_1$. □

Remark 3.4. (1) An analogue of Proposition 3.3(b) in [2; 4(iv)] is proved for the case when the σ-algebras $\mathcal{A}$ and $\mathcal{A}_1$ are both countably generated.

(2) A regular conditional probability p on $\mathcal{A}$ associated with μ relative to $\mathcal{A}_1$ is called *proper* if $p(A_1, x) = 1_{A_1}(x)$, $\forall x \in X$, $\forall A_1 \in \mathcal{A}_1$. In [25] it is shown that a regular conditional probability p on $\mathcal{A}$ associated with μ relative to $\mathcal{A}_1$ may exist, the σ-algebras $\mathcal{A}$ and $\mathcal{A}_1$ may be both countably generated, but a proper regular conditional probability p on A associated with μ relative to $\mathcal{A}_1$ may not exist. The proper regular conditional probabilities are studied in [26,27]; in [27] an example is given showing that Proposition 3.2(b) may not be true when $\mathcal{A}_1$ is not countably generated.

Proposition 3.5. *Let $(X, \mathcal{A}, \mu)$ be a probability space, $(Y, \mathcal{F})$ be a measurable space, $\eta: X \to Y$ be a $(\mathcal{A}, \mathcal{F})$-measurable map and $\mathcal{A}_\eta := \eta^{-1}(\mathcal{F})$.*

(a) *If a regular conditional probability q on $\mathcal{A}$ associated with μ relative to η exists, then a regular conditional probability p on $\mathcal{A}$ associated with μ relative to $\mathcal{A}_\eta$ exists as well and it can be obtained by the equality: $p(A, x) = q(A, \eta(x))$, $A \in \mathcal{A}$, $x \in X$.*

(b) If a regular conditional probability on $\mathcal{A}$ associated with μ relative to $\mathcal{A}_\eta$ exists and $\eta(X)$ is μ_η-measurable, then a regular conditional probability on $\mathcal{A}$ associated with μ relative to η exists as well.

Proof. (a) It is clear that p is a transition probability relative to $(X, \mathcal{A})$ and $(X, \mathcal{A}_\eta)$. Fix $A \in \mathcal{A}$ and $B \in \mathcal{F}$. By the change of variable formula we can write:

$$\int_{\eta^{-1}(B)} p(A, x)\, d\mu(x) = \int_B q(A, \eta(x))\, d\mu = \int_B q(A, y)\, d\mu_\eta(y),$$

By (MRCP3) we have:

$$\mu(A \cap \eta^{-1}(B)) = \int_B q(A, y)\, d\mu_\eta(y).$$

Consequently,

$$\mu(A \cap \eta^{-1}(B)) = \int_{\eta^{-1}(B)} p(A, x)\, d\mu(x).$$

Since $A \in \mathcal{A}$ and $B \in \mathcal{F}$ are arbitrary, the last equality means that p is a regular conditional probability associated with μ relative to $\mathcal{A}_\eta$.

(b) Let $p: \mathcal{A} \times X \to [0,1]$ be a regular conditional probability associated with μ relative to $\mathcal{A}_\eta$. Fix $A \in \mathcal{A}$. Since the function $p(A, \cdot)$ is $\mathcal{A}_\eta$-measurable, by Lemma 2.1, there exists a $\mathcal{F}$-measurable function $g_A: Y \to [0,1]$ such that $p(A, x) = g_A(\eta(x))$, $\forall x \in X$. By the assumption, $\eta(X)$ is μ_η-measurable. Therefore, there exists $Y_1 \in \mathcal{F}$ such that $Y_1 \subset \eta(X)$ and $\mu_\eta(Y_1) = 1$. Define a mapping $q: \mathcal{A} \times Y \to [0,1]$ by the equalities: $q(A, y) = g_A(y)$, $y \in Y_1$, $A \in \mathcal{A}$ and $q(A, y) = \mu(A)$, $y \in Y \setminus Y_1$, $A \in \mathcal{A}$.

Let us see now that q is a transition probability relative $(X, \mathcal{A})$ and $(Y, \mathcal{F})$.

For a fixed $A \in \mathcal{A}$ the function $q(A,\cdot)$ is $\mathcal{F}$-measurable, because g_A is and $Y_1 \in \mathcal{F}$.

Fix now $y \in Y_1$ and let us see that $q(\cdot, y)$ is a probability measure on $\mathcal{A}$. Since $y \in Y_1$ and $Y_1 \subset \eta(X)$, there exists $x \in X$ with $\eta(x) = y$. Then by the definition of q we have $q(A, y) = p(A,x), \ \forall A \in \mathcal{A}$. Since $p(\cdot, x)$ is a probability measure on $\mathcal{A}$, the set function $q(\cdot, y)$ is as well.

Note that q satisfies (MRCP3). Fix $A \in \mathcal{A}$ and $B \in \mathcal{F}$. As $\mu_\eta(Y_1) = 1$, we can write using the change of variable formula and (RCP3) for p:

$$\int_B q(A,y)\,d\mu_\eta(y) = \int_{B\cap Y_1} q(A,y)\,d\mu_\eta(y) = \int_{\eta^{-1}(B\cap Y_1)} q(A,\eta(x))d\mu(x) =$$

$$= \int_{\eta^{-1}(B\cap Y_1)} p(A,x)d\mu(x) = \mu(A\cap\eta^{-1}(B\cap Y_1)) = \mu(A\cap\eta^{-1}(B)).$$

Consequently, q is a regular conditional probability associated with μ relative to η. □

Remark 3.6. We note in connection with Proposition 3.5(b) that for a given measurable mapping the condition "$\eta(X)$ is μ_η-measurable" may not be satisfied even if $(Y, \mathcal{F})=(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. A probability space $(X, \mathcal{A}, \mu)$ (also the measure μ itself) is called *perfect*, if for every $\mathcal{A}$-measurable $\eta: X \to \mathbb{R}$ one has that $\eta(X)$ is μ_η-measurable. This concept was introduced in [28] and perfectness was even included as a postulate in the definition of a probability space. A substantial study of perfect measures was made in [29].

4. Disintegration of general probability measures.

In this section we clarify the relationship between regular conditional probabilities and disintegrations.

Let $(X, \mathcal{A}, \mu)$ be a probability space, $(Y, \mathcal{F})$ be a measurable space and $\eta: X \to Y$ be a $(\mathcal{A},\mathcal{F})$-measurable mapping. It is clear that a disintegration of μ with respect to η is a transition probability relative to $(X, \mathcal{A})$ and $(Y, \mathcal{F})$ with the additional properties (Dis3) and (Dis4).

Lemma 4.1. *Let $(X, \mathcal{A}, \mu)$ be a probability space, $(Y, \mathcal{F})$ be a measurable space and $\eta: X \to Y$ be a $(\mathcal{A},\mathcal{F})$-measurable mapping for which there exists a disintegration q of μ on $\mathcal{A}$ with respect to η. Then $\eta(X)$ is μ_η-measurable.*

Proof. Let N be as in (Dis3) and $Y_0 = Y \setminus N$. Then $Y_0 \in \mathcal{F}$ and $\mu_\eta(Y_0) = 1$. Fix arbitrarily $y \in Y_0$; since $q(\eta^{-1}(\{y\}, y)) = 1$, we get that $\eta^{-1}(\{y\}) \neq \varnothing$. Consequently, $y \in \eta(X)$ and hence, $Y_0 \subset \eta(X)$. The last inclusion (together with $Y_0 \in \mathcal{F}$ and $\mu_\eta(Y_0) = 1$) implies μ_η-measurability of $\eta(X)$. □

Proposition 4.2. *Let $(X, \mathcal{A}, \mu)$ be a probability space, $(Y, \mathcal{F})$ be a measurable space and $\eta: X \to Y$ be a $(\mathcal{A},\mathcal{F})$-measurable mapping.*

(a) (*cf.* [23, Theorem 2]) *If q is a regular conditional probability on $\mathcal{A}$ associated with μ relative to η and $gr(\eta) \in \mathcal{A} \otimes \mathcal{F}$, then q is a disintegration of μ on $\mathcal{A}$ with respect to η and $\eta(X)$ is μ_η-measurable.*

(b) *If q is a disintegration of μ on $\mathcal{A}$ with respect to η, then q is a regular conditional probability on $\mathcal{A}$ associated with μ, relative to η.*

(c) *If $\mathcal{A}$ is countably generated and q_1, q_1 are disintegrations of μ on $\mathcal{A}$ with respect to η, then there exists a set $N \in \mathcal{F}$ such that $\mu_\eta(N) = 0$ and*

$$q_1(A,y) = q_2(A,y), \quad \forall x \in Y \setminus N, \quad \forall A \in \mathcal{A}.$$

Proof. (a) Clearly, q has the properties (Disl), (Dis2) and (Dis4). It remains to show that q has the property (Dis3) as well.

Since $gr(\eta) \in \mathcal{A} \otimes \mathcal{F}$, we can apply Proposition 3.1 for $E = gr(\eta)$ and get

$$1=\mu\{x\in X:(x,\eta(x))\in E\}=\int_y q(E_{\cdot,y},y)\,d\mu_\eta(y)=\int_y q(\eta^{-1}(\{y\}),y)\,d\mu_\eta(y).$$

Hence, $\mu_\eta\{y\in Y:q(\eta^{-1}(\{y\}),y)=1\}=1$. Therefore, q is a disintegration and then μ_η-measurability of $\eta(X)$ follows from Lemma 4.1.

(b) (cf. [7, pp. 146-147]) Let $A\in\mathcal{A}$ and $B\in\mathcal{F}$, N be as in (Dis3) and $y\in Y/N$. Since $q(\eta^{-1}(\{y\}),y)=1$, we can write $q(A\cap\eta^{-1}(B),y)=q(A,y)1_B(y)$. As $\mu_\eta(N)=0$,

$$\int_B q(A,y)d\mu_\eta(y)=\int_{Y/N}q(A,y)1_B(y)d\mu_\eta(y)=\int_{Y/N}q(A\cap\eta^{-1}(B),y)\,d\mu_\eta(y)=\int_Y q(A\cap\eta^{-1}(B),y)d\mu_\eta(y).$$

i.e., $\int_B q(A,y)d\mu_\eta(y)=\int_Y q(A\cap\eta^{-1}(B),y)d\mu_\eta(y)$. From this and (Dis4) we get

$$\int_B q(A,y)\,d\mu_\eta(y)=\int_{Y/N}q(A,y)1_B(y)\,d\mu_\eta(y)=\int_Y q(A\cap\eta^{-1}(B),y)d\mu_\eta(y)=\mu(A\cap\eta^{-1}(B)).$$

Hence, q is a regular conditional probability on $\mathcal{A}$ associated with μ relative to η.

(c) By (b) we have that q_1, q_2 are regular conditional probabilities on $\mathcal{A}$ associated with μ relative to η. Since $\mathcal{A}$ is countably generated, the conclusion follows from Proposition 3.2.

Remark 4.3. We note:

(1) Proposition 4.2(a) is for $(Y,\mathcal{F})=(\mathbb{R},\mathcal{B}(\mathbb{R}))$ contained in [30, Lemma 3.1, p.290]; however, the proof presented there is not quite clear.

(2) Proposition 4.2(a) is not true in general (even if $\{y\}\in\mathcal{F},\ \forall y\in Y$). In fact, let $(X,\mathcal{A},\mu)=([0,1],\mathcal{B}([0,1]),\mu)$ (where μ stands for the Lebesgue measure), $(Y,\mathcal{F})=([0,1],\mathcal{F})$, where $\mathcal{F}$ is σ-algebra generated by finite subsets of $[0,1]$; put $\eta(x)=x,\ \forall x\in[0,1]$. Then $q:\mathcal{A}\times Y\to[0,1]$, defined by the equality $q(A,y)=\mu(A),\ A\in\mathcal{A},\ y\in Y$, is a regular conditional probability on $\mathcal{A}$ associated with μ relative to η. However, $q(\eta^{-1}(\{y\}),y)=0,\ \forall y\in Y$. Hence, q is not a disintegration of μ on $\mathcal{A}$ with respect to η.

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მათემატიკა

რეგულარული პირობითი ალბათობები და დეზინტეგრაციები

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