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Theory of Elasticity

Efficient Solution of BVPs of Thermoelasticity for Half-Plane

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ABSTRACT. In the present paper explicit solutions of first and second boundary value problems (BVP) of thermoelasticity are constructed for the two-dimensional equations of thermoelastic transversally isotropic half-plane. For their solutions we used the potential method and constructed special fundamental matrices, which reduced the first and second BVPs to a Fredholm integral equation of the second kind. *© 2007 Bull. Georg. Natl. Acad. Sci.*

Key words: *potential method, integral equations, thermoelasticity, fundamental matrix.*

Let D be the upper half-plane with the boundary $S(S: x_3 = 0)$, and the normal is (0,1).

We say that a body is subject to a plane deformation parallel to the plane Ox_1x_3 if the second component of the displacement vector $u(u_1, u_2, u_3)$ equals zero and the components u_1, u_3 depend only on x_1, x_3. In this case the basic two-dimensional equations of thermoelasticity for the transversally isotropic body can be written as follows [1]

$$C(\partial x)u = B\,\mathrm{grad}\,u_4, \tag{1}$$

$$\left(a_4 \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2}\right) u_4 = 0, \tag{2}$$

where $C(\partial x) = \left\| C_{pq}(\partial x) \right\|_{2\times 2}$, $B = \left\| B_{pq} \right\|_{2\times 2}$, $C_{11}(\partial x) = c_{11}\frac{\partial^2}{\partial x_1^2} + c_{44}\frac{\partial^2}{\partial x_3^2}$,

$C_{21}(\partial x) = C_{12}(\partial x) = (c_{13} + c_{44})\frac{\partial^2}{\partial x_1 \partial x_3}$, $C_{22}(\partial x) = c_{44}\frac{\partial^2}{\partial x_1^2} + c_{33}\frac{\partial^2}{\partial x_3^2}$, $B_{11} = \beta$, $B_{22} = \beta'$, $B_{12} = B_{12} = 0$,

$a_4 = \frac{k}{k'}$, $\beta = c_{13}\alpha' + 2\alpha(c_{11} - c_{66})$, $\beta' = c_{33}\alpha' + 2\alpha c_{13}$, α, α' are coefficients of temperature extension, k, k' are coefficients of thermal conductivity, c_{11}, c_{44}, c_{13}, c_{33} are Hooke's coefficients. $u = u(u_1, u_3)$ is a displacement vector, u_4 is the temperature of body.

Definition. The function $f(x)$ defined in D is called regular, if it has integrable in D continuous second derivatives and $f(x)$ itself and its first derivatives are continuously extendable at every point of S and the conditions of infinite are added $f(x)\in O(1)$, $\frac{\partial u}{\partial x_k}=O\left(|x|^{-2}\right)$, $k=1,3$, where $|x|^2=x_1^2+x_3^2$.

For the equation (1) -(2) we pose the following BVPs. Find a regular solution $u(x)$, $u_4(x)$, of the equations (1) -(2), if on the boundary S one of the following conditions are given:

Problem 1. $u^+=f(x_1)$, $u_4^+=f_4(x_1)$.

Problem 2. $\tau_{13}^+=c_{44}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right)=f_1(x_1)$, $\tau_{33}^+=c_{13}\frac{\partial u_1}{\partial x_1}+c_{33}\frac{\partial u_3}{\partial x_3}-\beta' u_4=f_3(x_1)$, $\frac{\partial u_4}{\partial x_3}=f_4(x_1)$.

From the equation (2) we find u_4 and the solution of the equation (1) will be presented in the form $u(x)=V(x)+u_0(x)$, where $V(x)$ is a solution of homogeneous equation $C(\partial x)V=0$, and $u_0(x)$ is a particular solution of equation $C(\partial x)u_0(x)=B\,\mathrm{grad}\,u_4$.

1. Solution to the first BVP for half-plane. A solution to the equation (2) in the domain D is

$$u_4(x)=\frac{1}{\pi}\mathrm{Im}\int_s\frac{f_4(t)}{t-z_4}dt, \tag{3}$$

where $z_4=x_1+i\sqrt{a_4}x_3$, $f_4\in H$.

One particular solution $u_0(x)$ to equation (1) is the following

$$u_0(x)=\frac{1}{\pi}\mathrm{Im}\sum_{k=2}^{4}\begin{Vmatrix}A_k & 0\\ 0 & B_k\end{Vmatrix}\int_s \mathrm{grad}\,\sigma_k\ln\sigma_k f_4(t)dt, \tag{4}$$

where $\sigma_k=t-(x_1+\alpha_k x_3)$, $\alpha_k=i\sqrt{a_k}$, $A_k=(-1)^k\left[A_4(c_{44}-c_{33}a_k)\sqrt{a_2a_3a_k^{-1}}+B_4\sqrt{a_4}(c_{13}+c_{44})\right]d$,

$B_k=(-1)^k\left[-A_4(c_{44}+c_{13})\sqrt{a_2a_3a_k^{-1}}+B_4\sqrt{a_4}\left(c_{44}-c_{33}a_2a_3a_k^{-1}\right)\right]d$, $d^{-1}=\left(\sqrt{a_2}-\sqrt{a_3}\right)\left(c_{44}+c_{33}\sqrt{a_2a_3}\right)$,

$d_4^{-1}=c_{33}c_{44}(a_4-a_2)(a_4-a_3)$, $A_4=[\beta(c_{44}-c_{33}a_4)+\beta' a_4(c_{13}+c_{44})]d_4$, $B_4=[-\beta(c_{44}+c_{13})+\beta'(c_{11}-c_{44}a_4)]d_4$,

a_k $(k=2,3)$ are the positive roots of a characteristic equation

$$c_{33}c_{44}a^2-\left(c_{11}c_{33}-c_{13}^2-2c_{13}c_{44}\right)a+c_{11}c_{44}=0.$$

It is easy to show that $u_0(x)=0$, when $x_3=0$.

A solution to the equation $C(\partial x)V=0$, when $V^+=f(t)$, will be sought in the domain D in terms of the double layer potential

$$V(x)=\frac{1}{\pi}\mathrm{Im}\sum_{k=2}^{3}\left\|N_{pq}^{(k)}\right\|_{2\times2}\int_s\frac{g(t)dt}{t-z_k}, \tag{5}$$

where

$$N_{11}^{(k)}=(-1)^k d(c_{33}a_k-c_{44})\sqrt{a_2a_3a_k^{-1}},\ N_{21}^{(k)}=i(-1)^k d(c_{13}+c_{44}),\ N_{12}^{(k)}=\sqrt{a_2a_3}N_{21}^{(k)},$$

$$N_{22}^{(k)}=(-1)^k d(c_{44}a_k-c_{11})\sqrt{a_k^{-1}},\ k=2,3.$$

$g(t)$ is an unknown real vector-function. To determine it we obtain the following integral equation

$$g(t_0)+\frac{1}{\pi}\operatorname{Im}\sum_{k=2}^{3}\left\|N_{pq}^{(k)}\right\|_{2\times 2}\int_s \frac{g(t)dt}{t-t_0}=f(t_0). \tag{6}$$

Taking into account the fact that $\sum_{k=2}^{3}N_{11}^{(k)}=\sum_{k=2}^{3}N_{11}^{(k)}=1$, $\sum_{k=2}^{3}N_{12}^{(k)}=0$. From the equation (6) we have $g(t_0)=f(t_0)$ and (5) takes the form

$$V(x)=\frac{1}{\pi}\operatorname{Im}\sum_{k=2}^{3}\left\|N_{pq}^{(k)}\right\|_{2\times 2}\int_s \frac{f(t)dt}{t-z_k}.$$

Thus we have obtained the Poisson type formula for the solution of the first BVP for the half-plane.

Note that $f\in C^{1,\alpha}(S)$ and satisfies the condition $f(t)=C+\frac{\alpha}{|t|^{1+\beta}}$ at infinity, where C and α are constant vectors and $\beta>0$.

2. Solution to the second BVP for half-plane. The solution to the equation (2) has the form

$$u_4(x)=\frac{1}{\pi\sqrt{a_4}}\operatorname{Re}\int_s \ln(t-z_4)f_4(t)\,dt.$$

A particular solution $u_0(x)$ to the equation (1), when $\tau_{13}(u_0)=0$, $\tau_{33}(u_0)=0$ on S, is the following vector

$$u_0(x)=\frac{1}{\pi}\operatorname{Re}\sum_{k=2}^{4}\begin{Vmatrix}A_k & 0\\ 0 & B_k\end{Vmatrix}\int_s \operatorname{grad}\sigma_k^2\ln\sigma_k f_4(t)\,dt,$$

where $A_k=(-1)^k c_{44}(c_{13}+c_{33}a_k)(A_4+B_4)\left(\sqrt{a_2a_3a_k^{-1}}-\sqrt{a_4}\right)n$,

$B_k=(-1)^k c_{44}(c_{11}+c_{13}a_k)(A_4+B_4)\left(\sqrt{a_2a_3a_k^{-1}}-\sqrt{a_4}\right)a_k^{-1}n$,

$n^{-1}=\left(\sqrt{a_2}-\sqrt{a_3}\right)\left(c_{11}c_{33}-c_{13}^2\right)$, $s^{-1}=2c_{33}c_{44}\sqrt{a_4}(a_4-a_2)(a_4-a_3)$, $A_4=[\beta(c_{44}-c_{33}a_4)+\beta' a_4(c_{13}+c_{44})]s$,
$B_4=[-\beta(c_{44}+c_{13})+\beta'(c_{11}-c_{44}a_4)]s$.

Now let us consider the second BVP for the equation $C(\partial x)V=0$. We look for the solution as a single layer potential of the second kind

$$V(x)=\frac{1}{\pi}\operatorname{Re}\sum_{k=2}^{4}\left\|L_{pq}^{(k)}\right\|_{2\times 2}\int_s \ln(t-z_k)h(t)dt, \tag{7}$$

where h is an unknown real vector; the coefficients $L_{pq}^{(k)}$ can be written as follows:

$$L_{11}^{(k)}=(-1)^k(c_{13}+c_{33}a_k)n,\quad L_{12}^{(k)}=(-1)^k i(c_{13}+c_{33}a_k)\sqrt{a_2a_3a_k^{-1}}\,n,$$

$$L_{21}^{(k)}=(-1)^k i(c_{11}+c_{13}a_k)\sqrt{a_k^{-1}}\,n,\quad L_{22}^{(k)}=(-1)^{k+1}(c_{11}+c_{13}a_k)a_k^{-1}\sqrt{a_2a_3}\,n.$$

Taking into account the boundary condition $\tau_{13}^{+}(V)=f_1(x_1)$, $\tau_{33}^{+}(V)=c_{13}\dfrac{\partial V_1}{\partial x_1}+c_{33}\dfrac{\partial V_3}{\partial x_3}=f_3(x_1)$, $x_1\in S$, after direct calculation we find $h(t)=f(t)$.

Therefore, we have the following Poisson type formula for the solution of the second BVP

$$V(x)=\frac{1}{\pi}\operatorname{Re}\sum_{k=2}^{4}\left\|L_{pq}^{(k)}\right\|_{2\times 2}\int_{s}\ln\left(t-z_k\right)f(t)dt.$$

For the regularity of the solution $V(x)$ it is sufficient that $\int_{s} f(t)dt=0$, $f\in C^{0,\alpha}(S)$, $\alpha>0$, and $f(t)=O\left(\left|t^{-1-\beta}\right|\right)$, $\beta>0$, for large $|t|$.

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