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Mathematics

Equivalence of Convergence for Almost all Signs and Almost all Rearrangements of Functional Series

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ABSTRACT. It is well known that the convergence of the series $\sum_1^\infty a_l\theta_l$ in a Banach space for all sequences of signs (θ_l) is equivalent to the convergence of all rearrangements of $\sum_1^\infty a_l$. We find an analogue of this fact in the case when instead of the convergence for *all* signs we have only the convergence for *almost all* signs. The results make sense even in the scalar case. We also find an application of the result to the following Nikishin problem: Assume a series $\sum_1^\infty \xi_l$ of random variables is such that a subsequence of partial sums tends to a random variable *S*. When does there exist a rearrangement of the series convergent to S almost surely? *© 2009 Bull. Georg. Natl. Acad. Sci.*

Key words: *convergence of series, almost all signs, almost all simple permutations, rearrangement convergence almost surely.*

1. Introduction

Let $\sum_1^\infty a_l$ be a series with terms in a Banach space X convergent to an element $S \in X$. It is well known that the series $\sum_1^\infty a_l\theta_l$ converges for all sequences $\theta = (\theta_1, \theta_2, ...)$ of signs if and only if the series $\sum_1^\infty a_{\pi\{l\}}$ converges for all permutations $\pi : N \to N$. The main question we consider in this note is: what happens, if instead of convergence for *all* signs we require the convergence for *almost all* signs, i.e. the convergence almost surely of $\sum_1^\infty a_l r_l$, where (r_l) is a sequence of Rademacher random variables taking values ± 1 with equal probabilities? Or, in other words, what is the permutational counterpart of the convergence of $\sum_1^\infty a_l\theta_l$ for almost all signs $\theta = (\theta_l)$? We show that the corresponding condition expressed in terms of permutations can be stated as follows: The series $\sum_1^\infty a_{\pi\{l\}}$ converges *for almost all simple permutations* $\pi : N \to N$, where the latter notion is appropriately understood.

Our results have applications to the following problem initiated by Garsia [1] and Nikishin [2]. Let (ξ_l) be a sequence of random variables such that a sequence of partial sums $S_{k_n} = \sum_1^{k_n} \xi_l$ converges a.s. to a random variable

S. When does there exist a permutation $\pi : N \to N$ such that the series $\sum_1^\infty \xi_{\pi(l)}$ converges a.s. to S ? We prove in particular that this is the case, if $\sum_{k_n+1}^{k_{n+1}} \xi_l r_l$ goes a.s. to zero, where (r_l) is the Rademacher sequence independent of (ξ_l). Besides, we prove that under this condition the set of permutations π ensuring the convergence a.s. of $\sum_1^\infty \xi_{\pi(l)}$ to S is rich enough. For some classes of Banach spaces the results can be expressed through the individual summands a_l -s and ξ_l -s. These results improve and generalize the known results.

Nikishin type theorems suggest the existence of a series that converges in measure but none of its rearrangements converges a.s. We have constructed such an example with an additional condition of convergence to zero of the general term. The example will be published separately.

2. Notations.

$(\Omega, \mathcal{A}, P)$ denotes underlying probability space. A mapping $\xi : \Omega \to X$ is said to be a random variable in a Banach space X (an X -valued random variable), if it is Bochner measurable. Let $k = (k_n)$, $1 = k_1 < k_2 < \ldots$ be a sequence of integers, $I_n^k = \{k_n + 1, \ldots, k_{n+1}\}$, $n \in N$ be the sequence of corresponding blocks, Π_n^k be the group of all permutations $\pi^{(n)}$: $I_n^k \to I_n^k$, μ_n^k be the uniform probability distribution on Π_n^k (assigning to each $\pi^{(n)}$ the probability $1/(k_{n+1} - k_n)!$), and let Σ_n^k be the σ -algebra of all subsets of Π_n^k. Then we consider the product of probability spaces

$$(\Pi^k, \Sigma^k, \mu^k) = \prod_{n=1}^\infty (\Pi_n^k, \Sigma_n^k, \mu_n^k).$$

Note that Π^k is a Tikhonov compact group and μ^k is the Haar measure on it. Each element $(\pi^{(1)}, \pi^{(2)}, \ldots) \in \Pi^k$ defines a permutation $\pi : N \to N$. Namely, if $l \in I_n^k$, then $\pi(l) = \pi^{(n)}(l) \in I_n^k$. We say that such a π is a *simple* permutation acting within the blocks I_n^k, $n \in N$, or that I_n^k, $n \in N$ are invariant blocks for π.

3. Convergence of series for almost all permutations. The case of constant summands.

Let us start with the case when the summands are constants.

Theorem 1. *Let (a_k) be a sequence of elements of a Banach space X such that a sequence of partial sums $S_{k_n} = \sum_1^{k_n} a_l$, $n \in N$, converges to $S \in X$. In order that $\sum_1^\infty a_{\pi(l)}$ converges to S for μ^k -almost all π's it is necessary and sufficient that*

$$\sum_{k_n+1}^{k_{n+1}} a_l r_l \to 0 \quad \lambda\text{-almost surely} \tag{1}$$

where $(r_n), n \in N$ is the sequence of Rademacher functions defined on $[0,1]$ *with the Lebesgue measure λ and* $k = (k_n)$, $1 = k_1 < k_2 < \ldots$.

Proof of the theorem is based on the following two-sided inequality found in [3].

Lemma 1. *Let $x_1, \ldots, x_n$ be elements of a normed space X, real or complex, $S = \sum_1^n x_l$. Then for each $t > 0$ the following two-sided inequality holds*:

$$\begin{aligned} \lambda\{u : \| \sum_1^n x_l r_l(u) \| > 2t + \| S \|\} \le \frac{1}{n!}\, card\{\pi : \max_{k \le n} \| \sum_1^k x_{\pi(l)} \| > t - \| S \|\} \le \\ 10\lambda\{u : \| \sum_1^n x_l r_l(u) \| > \frac{t}{12} - 2\| S \|\}. \end{aligned} \tag{2}$$

Lemma 1 goes back to the monograph [4] where a version of the right-hand-side inequality was found for X = **R**. Inequality (2) generalizes the Maurey-Pisier theorem [5] stating that for some absolute positive constants C_1 and C_2 the following two-sided inequality holds

$$C_1 E \| \sum_1^n x_l r_l \| \le \frac{1}{n!} \sum_\pi \max_{k \le n} \| \sum_1^k x_{\pi(l)} \| \le C_2 E \| \sum_1^n x_l r_l \|.$$

For another generalization see [6] where a further development of the inequality is conjectured and proved for $X = \mathbf{R}$.

Let us sketch the proof of Theorem 1. Denote $V_n = \sum_{k_n+1}^{k_{n+1}} a_l$ and $M_n^\pi = \max_{k_n < m \le k_{n+1}} \| \sum_{k_n+1}^m a_{\pi(l)} \|$. To prove the sufficiency part we use the right-hand-side of (2) to get that for any $\varepsilon > 0$

$$\sum_1^\infty \mu^k \{\pi : M_n^\pi + \| V_n \| > \varepsilon\} \le 10 \sum_{n=1}^\infty \lambda \{u : \| \sum_{k_n+1}^{k_{n+1}} a_l r_l(u) \| + 2 \| V_n \| > \frac{\varepsilon}{12}\}. \tag{3}$$

Convergence of S_{k_n} implies the convergence of V_n to zero. Therefore, condition (1), independence of the Rademacher sequence (r_l) and the necessity part of the Borel-Cantelli lemma imply the finiteness of the right-hand-side of (3). Hence, the left-hand-side of (3) is also finite. Now, due to the sufficiency part of the Borel-Cantelli lemma the latter implies the convergence μ^k-a.s. to zero of $M_n^\pi + \| V_n \|$ and hence the convergence μ^k-a.s. to zero of M_n^π.

We have due to the left-hand-side part of (2): for any $\varepsilon > 0$

$$\sum_{n=1}^\infty \lambda\{u : \| \sum_{k_n+1}^{k_{n+1}} a_l r_l(u) \| - \| V_n \| > \varepsilon\} \le \sum_1^\infty \mu^k \{\pi : M_n^\pi + \| V_n \| > \frac{\varepsilon}{2}\} < \infty. \tag{4}$$

Since $M_n^\pi \to 0$ $\mu^k - a.s.$ and $V_n \to 0$, (4) implies (1).

4. Condition (1) expressed in terms of coefficients.

In the light of Theorem 1 of interest are conditions ensuring condition (1) in terms of coefficients (a_l). A rich source of such conditions is provided by the fact that (1) is implied by the convergence a.s. of $\sum_1^\infty a_l r_l$.

Theorem 2. *Each of the following conditions (i), (ii) and (iii) implies the convergence a.s. of* $\sum_1^\infty a_l r_l$, *and therefore implies condition (1) for any* $k = (k_n),\ k_1 = 1,\ k_1 < k_2, \ldots$ *; Under the conditions of Theorem 1 each of the conditions (i), (ii) and (iii) ensure the convergence of* $\sum_1^\infty a_{\pi(l)}$ *to* S *for* μ^k*-almost all* π*-s.*

(i) X *is a general Banach space and* $\sum_1^\infty \rho(\| a_l \|) < \infty$ *, where* ρ *is the modulus of smoothness of* X *;*

(ii) X *is a Banach lattice of some cotype* q, $2 \le q < \infty$ *and*

$$(\sum_1^n |a_l|^2)^{\frac{1}{2}} \text{ converges in } X \text{ as } n \to \infty;$$

(iii) $X = L_p(T, \Sigma, \nu)$, $1 \le p < \infty$,

where the measure ν *is* σ*-finite, and*

$$\int_T (\sum_1^\infty |a_l(t)|^2)^{\frac{p}{2}} d\nu(t) < \infty .$$

The fact that the convergence of $\sum_1^\infty a_l r_l$ follows from (i) was proved in [7] ; that it follows from (ii) was proved in [8]; and that (iii) coincides with (ii) for L_p-spaces, $1 \le p < \infty$, can also be found in [8].

5. The existence setting in the case of constant summands.

In the 1960s and 70s the problem of *existence* of a permutation ensuring the convergence of a series was very popular. It is closely related to the famous problem on the structure of the sum range of a conditionally convergent series in finite-dimensional and infinite-dimensional spaces (see the monograph [9]). Obviously Theorems 1 and 2 give as corollaries some existence conditions. All the existence results listed below follow from these theorems. For additional information the reader is referred to [9,7,3]. The first result for the infinite-dimensional case was found by M.I.Kadets [10] where he has shown the existence of the desired permutation under the condition $\sum_1^\infty \| a_l \|^d < \infty$, where $d = \min(2, p)$. Later on his students have shown in [11,12] the existence of the permutation under condition (i) of Theorem 2 for a general Banach space. Then in [13] much weaker condition (iii) of Theorem 2 in the existence setting was found for L_p -spaces, true, only for $1 \le p \le 2$ (as we know from Theorem 2, the result holds for all p, $1 \le p < \infty$). Further, as we already noticed, Theorem 1 implies the sufficiency of (1) and therefore, the sufficiency of the convergence of $\sum_1^\infty a_l r_l$ for the existence of a desired permutation. This fact which is considered to be the most effective general condition in the infinite-dimensional case was established in [5], and independently in [3,7] by different methods. For the sake of completeness of presentation let us give the following strongest existence result which does not follow from Theorems 1 or 2.

Theorem 3. *Let* $\sum_1^\infty a_l$ *be a series in a normed space* X *such that* $S_{k_n} = \sum_1^{k_n} a_l$, $n \in N$, *converges to* $S \in X$. *Then there is a permutation* $\pi : N \to N$ *such that* $\sum_1^\infty a_{\pi(l)} = S$ *provided that the following* σ - θ *-condition is satisfied:*

For any permutation $\sigma : N \to N$ *there is a sequence of signs* $\theta = (\theta_1, \theta_2, ...)$ *such that the series* $\sum_1^\infty a_{\sigma(l)} \theta_l$ *converges in* X.

Theorem 3 was proved in [14] and independently by use of a different method in [15]. Obviously, the σ - θ - condition is weaker than the convergence of $\sum_1^\infty a_l r_l$. Moreover, in the finite-dimensional case the σ - θ - condition is satisfied, if just $a_l \to 0$ as $l \to \infty$. Hence, Theorem 3 for a finite-dimensional X gives the Steinitz theorem [16] saying that Theorem 3 holds true, if $a_l \to 0$ as $l \to \infty$. However, convergence of $\sum_1^\infty a_l r_l$, although much stronger than the σ - θ -condition, according to Theorem 1, ensures more: convergence of $\sum_1^\infty a_{\pi(l)}$ for μ^k - almost all π -s.

6. Equivalence between the convergence of series for almost all signs and almost all permutations.

Let $\sum_1^\infty a_l$ be a convergent series in a normed space X. We say that it converges for *almost all simple permutations*, if for each sequence $k = (k_n)$, $k_1 = 1$, $k_1 < k_2 <, ...$

$$\mu^k \{ \pi \in \Pi^k : \sum_1^\infty a_{\pi(l)} \quad converges \} = 1 .$$

Applying Theorem 1 to each sequence of the partial sums of a convergent series we come to the following assertion.

Theorem 4. *Let* $\sum_1^\infty a_l$ *be a convergent series in a Banach space* X. *The following are equivalent.*

(i) $\sum_1^\infty a_l r_l$ *converges a.s. in* X ;

(ii) $\sum_1^\infty a_l$ *converges in* X *for almost all simple permutations.*

7. Random series: Convergence almost surely for almost all permutations.

Here we apply Theorem 1 to series of random variables. By virtue of the Fubini theorem we can state the following corollary to Theorem 1.

Theorem 5. *Let* (ξ_k) *be a sequence of random variables defined on* $(\Omega, \mathcal{A}, P)$ *and taking values in a normed space* X *such that a sequence of partial sums* $S_{k_n} = \sum_1^{k_n} \xi_l$, $n \in N$, *converges* P *-a.s. to an* X *-valued random variable* S . *In order that* $\sum_1^\infty \xi_{\pi(l)}$ *converges to* S $\mu^k \times P$ *-almost surely, it is necessary and sufficient that*

$$\sum_{k_n+1}^{k_{n+1}} \xi_l r_l \to 0 \quad \lambda \times P\text{-almost surely} \tag{5}$$

where (r_n), $n \in N$ *is the sequence of Rademacher functions defined on* $[0,1]$ *with the Lebesgue measure* λ *and* $k = (k_n)$, $1 = k_1 < k_2 < \ldots$.

Let us remark as in the case of constants in Section 3 that the condition

$$\sum_1^\infty \xi_l r_l \text{ converges } \lambda \times P\text{-almost surely} \tag{6}$$

is stronger than (5) and therefore implies the $\mu^k \times P$ -almost sure convergence of $\sum_1^\infty \xi_{\pi(l)}$. The condition was stated in [17] as an existence condition. Although not necessary, (6) proves to be convenient especially when the sequence $k = (k_n)$ is not known. Another benefit of (6) is that for the classes of Banach spaces, in contrast to (5), it can be expressed effectively in terms of individual summands ξ_l -s (see Section 3). We don't give here all the corollaries, instead we restrict ourselves with the case of scalar ξ_l -s that leads in the existence setting to the famous Nikishin and Garsia theorems.

Corollary. (a) (Nikishin, [2].) *Assume a series* $\sum_1^\infty \xi_l$ *of real or complex random variables converges in measure and* $\sum_1^\infty |\xi_l|^2 < \infty$ *a.s. Then there exists a permutation* $\pi : N \to N$ *such that* $\sum_1^\infty \xi_{\pi(l)}$ *converges a.s.*

(b) (Garsia, [1].) *Let* $(\varphi_l) \subset L_2(T, \Sigma, \nu)$ *be an orthonormal system and* (α_l) *be real or complex coefficients with* $\sum_1^\infty |\alpha_l|^2 < \infty$. *Then there exists a permutation* $\pi : N \to N$ *such that* $\sum_1^\infty \alpha_{\pi(l)} \varphi_{\pi(l)}$ *converges a.s.*

In [18] we have found a simple straightforward way of proving the Garsia inequality that leads to Corollary (b).

The method based on inequality (2) we used in this paper can be applied to different areas of probability and analysis to get the existence or massiveness of a.s. convergent rearrangements of normalized sequences. In this way we have found various formulations of the strong laws of large numbers under rearrangements [19, 20].

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მათემატიკა

ფუნქციონალურ მწკრივთა კრებადობის ეკვივალენტურობა თითქმის ყველა ნიშნისა და თითქმის ყველა გადანაცვლებისათვის

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(წარმოდგენილია აკადემიკოს ნ. ვახანიას მიერ)

ცნობილია, რომ $\sum_1^\infty a_l\theta_l$ მწკრივის კრებადობა ბანახის სივრცეში (θ_l) ყველა ნიშნების მიმდევრობისათვის ეკვივალენტურია $\sum_1^\infty a_l$ მწკრივის კრებადობისა ყველა გადანაცვლებისათვის. ჩვენ ვპოულობთ ამ ფაქტის ანალოგს იმ შემთხვევისათვის, როცა ნიშანთა ყველა მიმდევრობის ნაცვლად ჩვენ გვაქვს კრებადობა *თითქმის ყველა* ნიშანთა მიმდევრობისათვის. მიღებული შედეგები ახალია სკალარულ შემთხვევაშიც. ჩვენ ვიყენებთ მიღებულ შედეგებს ე.ნიკიშინის შემდეგი ამოცანის გამოსაკვლევად: დავუშვათ შემთხვევით სიდიდეთა $\sum_1^\infty \xi_l$ მწკრივის კერძო ჯამების ქვემიმდევრობა იკრიბება S შემთხვევითი სიდიდისაკენ. რა შეზღუდვებით არსებობს მწკრივის S-კენ კრებადი გადანაცვლება?

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