Mathematics

Equivalence of Convergence for Almost all Signs and Almost all Rearrangements of Functional Series

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ABSTRACT. It is well known that the convergence of the series \( \sum_{i=1}^{\infty} a_i \theta_i \) in a Banach space for all sequences of signs \((\theta_i)\) is equivalent to the convergence of all rearrangements of \( \sum_{i=1}^{\infty} a_i \). We find an analogue of this fact in the case when instead of the convergence for all signs we have only the convergence for almost all signs. The results make sense even in the scalar case. We also find an application of the result to the following Nikishin problem: Assume a series \( \sum_{i=1}^{\infty} \xi_i \) of random variables is such that a subsequence of partial sums tends to a random variable \( S \). When does there exist a rearrangement of the series convergent to \( S \) almost surely? © 2009 Bull. Georg. Natl. Acad. Sci.

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1. Introduction

Let \( \sum_{i}^{\infty} a_i \) be a series with terms in a Banach space \( X \) convergent to an element \( S \in X \). It is well known that the series \( \sum_{i}^{\infty} a_i \theta_i \) converges for all sequences \( \theta = (\theta_1, \theta_2, \ldots) \) of signs if and only if the series \( \sum_{i}^{\infty} a_{\pi(i)} \) converges for all permutations \( \pi : N \to N \). The main question we consider in this note is: what happens, if instead of convergence for all signs we require the convergence for almost all signs \( \theta = (\theta_i) \)?

We show that the corresponding condition expressed in terms of permutations can be stated as follows: The series \( \sum_{i}^{\infty} a_{\pi(i)} \) converges for almost all simple permutations \( \pi : N \to N \), where the latter notion is appropriately understood.

Our results have applications to the following problem initiated by Garsia [1] and Nikishin [2]. Let \( (\xi_i) \) be a sequence of random variables such that a sequence of partial sums \( S_{k_n} = \sum_{i=1}^{k_n} \xi_i \) converges a.s. to a random variable

When does there exist a permutation $\pi : N \to N$ such that the series $\sum_{i=1}^{\infty} \xi_{\pi(i)}$ converges a.s. to $S$? We prove in particular that this is the case, if $\sum_{k=1}^{\infty} \xi_{l_k} r_l$ goes a.s. to zero, where $(r_l)$ is the Rademacher sequence independent of $(\xi_l)$. Besides, we prove that under this condition the set of permutations $\pi$ ensuring the convergence a.s. of $\sum_{i=1}^{\infty} \xi_{\pi(i)}$ to $S$ is rich enough. For some classes of Banach spaces the results can be expressed through the individual summands $a_i$-s and $\xi_l$-s. These results improve and generalize the known results.

Nikishin type theorems suggest the existence of a series that converges in measure but none of its rearrangements converges a.s. We have constructed such an example with an additional condition of convergence to zero of the general term. The example will be published separately.

2. Notations.

$(\Omega, A, P)$ denotes underlying probability space. A mapping $\xi : \Omega \to X$ is said to be a random variable in a Banach space $X$ (an $X$-valued random variable), if it is Bochner measurable. Let $k = (k_n), 1 = k_1 < k_2 < \ldots$ be a sequence of integers, $I_n^k = \{k_n + 1, \ldots, k_{n+1}\}, n \in N$ be the sequence of corresponding blocks, $\Pi_n^k$ be the group of all permutations $\pi^{(n)} : I_n^k \to I_n^k$, $\mu_n^k$ be the uniform probability distribution on $\Pi_n^k$ (assigning to each $\pi^{(n)}$ the probability $1/(k_{n+1} - k_n)!$), and let $\Sigma_n^k$ be the $\sigma$-algebra of all subsets of $\Pi_n^k$. Then we consider the product of probability spaces $(\Pi_n^k, \Sigma_n^k, \mu_n^k) = \prod_{n=1}^{\infty} (\Pi_n^k, \Sigma_n^k, \mu_n^k)$.

Note that $\Pi_n^k$ is a Tikhonov compact group and $\mu_n^k$ is the Haar measure on it. Each element $\pi \in \Pi_n^k$ defines a permutation $\pi : N \to N$. Namely, if $l \in I_n^k$, then $\pi(l) = \pi^{(n)}(l) \in I_n^k$. We say that such a $\pi$ is a simple permutation acting within the blocks $I_n^k, n \in N$, or that $I_n^k, n \in N$ are invariant blocks for $\pi$.

3. Convergence of series for almost all permutations. The case of constant summands.

Let us start with the case when the summands are constants.

**Theorem 1.** Let $(a_k)$ be a sequence of elements of a Banach space $X$ such that a sequence of partial sums $S_{k_n} = \sum_{i=1}^{k_n} a_i, n \in N$, converges to $S \in X$. In order that $\sum_{i=1}^{\infty} a_{\pi(i)}$ converges to $S$ for $\mu^k$-almost all $\pi$’s it is necessary and sufficient that

$$\sum_{k=1}^{\infty} a_k r_k l_k \to 0 \quad \lambda - \text{almost surely}$$

where $(r_n), n \in N$ is the sequence of Rademacher functions defined on $[0,1]$ with the Lebesgue measure $\lambda$ and $k = (k_n), 1 = k_1 < k_2 < \ldots$.

Proof of the theorem is based on the following two-sided inequality found in [3].

**Lemma 1.** Let $x_1, \ldots, x_n$ be elements of a normed space $X$, real or complex, $S = \sum_{i=1}^{n} x_i$. Then for each $t > 0$ the following two-sided inequality holds:

$$\lambda(u : \| \sum_{i=1}^{n} x_i r_i (u) \| > 2t \| S \|) \leq \frac{1}{n!} \text{card} \{ \pi : \max_{1 \leq k \leq n} \| \sum_{i=1}^{n} x_{\pi(i)} \| > t - \| S \| \} \leq 10 \lambda(u : \| \sum_{i=1}^{n} x_i r_i (u) \| > \frac{t}{12} - 2 \| S \|).$$

Lemma 1 goes back to the monograph [4] where a version of the right-hand-side inequality was found for $X = \mathbb{R}$. Inequality (2) generalizes the Maurey-Pisier theorem [5] stating that for some absolute positive constants $C_1$ and $C_2$ the following two-sided inequality holds.

Equivalence of Convergence for Almost all Signs and Almost all Rearrangements of Functional Series


$$C_1 E \| \sum_{i=1}^{n} x_i r_i \| \leq \frac{1}{n} \sum_{i} \max_{k \leq n} \| \sum_{k} x_{\pi(i)} \| \leq C_2 E \| \sum_{i} x_i r_i \|.$$  

For another generalization see [6] where a further development of the inequality is conjectured and proved for $X = \mathbb{R}$.

Let us sketch the proof of Theorem 1. Denote $S_{k_n}$ and $\max_{1 \leq k \leq n} \| x_{\pi(i)} \|$.

To prove the sufficiency part we use the right-hand-side of (2) to get that for any $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \mu^k (\pi : M_n^x + \| V_n \| > \varepsilon) \leq 10 \sum_{n=1}^{\infty} \lambda (\| \sum_{k=1}^{n} a_k r_k(u) \| + \| V_n \| > \varepsilon / 2).$$  

(3)

Convergence of $S_{k_n}$ implies the convergence of $V_0$ to zero. Therefore, condition (1), independence of the Rademacher sequence $(r_i)$ and the necessity part of the Borel-Cantelli lemma imply the finiteness of the right-hand-side of (3). Hence, the left-hand-side of (3) is also finite. Now, due to the sufficiency part of the Borel-Cantelli lemma the latter implies the convergence $\mu^k$-a.s. to zero of $M_n^x + \| V_n \|$ and hence the convergence $\mu^k$-a.s. to zero of $M_n^\pi$.

We have due to the left-hand-side part of (2): for any $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \lambda (\| \sum_{k=1}^{n} a_k r_k(u) \| + \| V_n \| > \varepsilon) \leq \sum_{n=1}^{\infty} \mu^k (\pi : M_n^x + \| V_n \| > \varepsilon / 2) < \infty.$$

(4)

Since $M_n^\pi \to 0 \mu^k$-a.s. and $V_n \to 0$, (4) implies (1).

4. Condition (1) expressed in terms of coefficients.

In the light of Theorem 1 of interest are conditions ensuring condition (1) in terms of coefficients $(a_i)$. A rich source of such conditions is provided by the fact that (1) is implied by the convergence a.s. of $\sum_{i} a_i r_i$.

Theorem 2. Each of the following conditions (i), (ii) and (iii) implies the convergence a.s. of $\sum_{i} a_i r_i$, and therefore implies condition (1) for any $k = (k_n)$, $k_1 = 1, k_1 < k_2, \ldots$ ; Under the conditions of Theorem 1 each of the conditions (i), (ii) and (iii) ensure the convergence of $\sum_{i} a_{\pi(i)}$ to $S$ for $\mu^k$-almost all $\pi$-s.

(i) $X$ is a general Banach space and $\sum_{i} \rho (a_i) < \infty$, where $\rho$ is the modulus of smoothness of $X$ ;

(ii) $X$ is a Banach lattice of some cotype $q$, $2 \leq q < \infty$ and

$$\left( \sum_{i} a_i^2 \right)^{1/2} converges in X as n \to \infty ;$$

where the measure $\nu$ is $\sigma$-finite, and

$$\int_{T} \left( \sum_{i} a_i(t)^2 \right)^{p/2} d\nu(t) < \infty .$$

The fact that the convergence of $\sum_{i} a_i r_i$ follows from (i) was proved in [7] ; that it follows from (ii) was proved in [8]; and that (iii) coincides with (ii) for $L_p$-spaces, $1 \leq p < \infty$, can also be found in [8].

5. The existence setting in the case of constant summands.

In the 1960s and 70s the problem of existence of a permutation ensuring the convergence of a series was very popular. It is closely related to the famous problem on the structure of the sum range of a conditionally convergent series in finite-dimensional and infinite-dimensional spaces (see the monograph [9]). Obviously Theorems 1 and 2
give as corollaries some existence conditions. All the existence results listed below follow from these theorems. For additional information the reader is referred to [9,7,3]. The first result for the infinite-dimensional case was found by M.I. Kadets [10] where he has shown the existence of the desired permutation under the condition \( \sum_{i=1}^{\infty} \| a_i \| < \infty \), where \( d = \min (2, p) \). Later on his students have shown in [11,12] the existence of the permutation under condition (i) of Theorem 2 for a general Banach space. Then in [13] much weaker condition (iii) of Theorem 2 in the existence setting was found for \( L_p \)-spaces, true, only for \( 1 \leq p \leq 2 \) (as we know from Theorem 2, the result holds for all \( p, 1 \leq p < \infty \)). Further, as we already noticed, Theorem 1 implies the sufficiency of (1) and therefore, the sufficiency of the convergence of \( \sum_{i=1}^{\infty} a_i r_i \) for the existence of a desired permutation. This fact which is considered to be the most effective general condition in the infinite-dimensional case was established in [5], and independently in [3,7] by different methods. For the sake of completeness of presentation let us give the following strongest existence result which does not follow from Theorems 1 or 2.

**Theorem 3.** Let \( \sum_{n=1}^{\infty} a_n \) be a series in a normed space \( X \) such that \( S_{k_n} = \sum_{i=1}^{k_n} a_i, \ n \in N \), converges to \( S \in X \). Then there is a permutation \( \pi : N \rightarrow N \) such that \( \sum_{i=1}^{\infty} a_{\pi(i)} = S \) provided that the following \( \sigma - \theta \)-condition is satisfied:

For any permutation \( \sigma : N \rightarrow N \) there is a sequence of signs \( \theta = (\theta_1, \theta_2, \ldots) \) such that the series \( \sum_{i=1}^{\infty} a_{\pi(i)} \theta_i \) converges in \( X \).

Theorem 3 was proved in [14] and independently by use of a different method in [15]. Obviously, the \( \sigma - \theta \)-condition is weaker than the convergence of \( \sum_{i=1}^{\infty} a_i r_i \). Moreover, in the finite-dimensional case the \( \sigma - \theta \)-condition is satisfied, if just \( a_i \rightarrow 0 \) as \( l \rightarrow \infty \). Hence, Theorem 3 for a finite-dimensional \( X \) gives the Steinitz theorem [16] saying that Theorem 3 holds true, if \( a_i \rightarrow 0 \) as \( l \rightarrow \infty \). However, convergence of \( \sum_{i=1}^{\infty} a_i r_i \), although much stronger than the \( \sigma - \theta \)-condition, according to Theorem 1, ensures more: convergence of \( \sum_{i=1}^{\infty} a_{\pi(i)} \) for \( \mu^k \)-almost all \( \pi \)-s.

### 6. Equivalence between the convergence of series for almost all signs and almost all permutations.

Let \( \sum_{n=1}^{\infty} a_n \) be a convergent series in a normed space \( X \). We say that it converges for **almost all simple permutations**, if for each sequence \( k = (k_n), \ k_1 = 1, \ k_i < k_{i+1} \),

\[
\mu^k \{ \pi \in \Pi^k : \sum_{i=1}^{\infty} a_{\pi(i)} \text{ converges} \} = 1.
\]

Applying Theorem 1 to each sequence of the partial sums of a convergent series we come to the following assertion.

**Theorem 4.** Let \( \sum_{n=1}^{\infty} a_n \) be a convergent series in a Banach space \( X \). The following are equivalent.

(i) \( \sum_{n=1}^{\infty} a_n r_n \) converges a.s. in \( X \);

(ii) \( \sum_{n=1}^{\infty} a_n \) converges in \( X \) for almost all simple permutations.

### 7. Random series: Convergence almost surely for almost all permutations.

Here we apply Theorem 1 to series of random variables. By virtue of the Fubini theorem we can state the following corollary to Theorem 1.
Theorem 5. Let \((\xi_k)\) be a sequence of random variables defined on \((\Omega, A, P)\) and taking values in a normed space \(X\) such that a sequence of partial sums \(S_k = \sum_{i=1}^{k} \xi_i\), \(n \in N\), converges \(P\)-a.s. to an \(X\)-valued random variable \(S\). In order that \(\sum_{l=1}^{\infty} \xi_{\pi(l)}\) converges to \(S\) \(\mu \times P\)-almost surely, it is necessary and sufficient that

\[\sum_{k=1}^{\infty} \sum_{j=1}^{k} r_j = 0\]

where \((r_n), n \in N\) is the sequence of Rademacher functions defined on \([0,1]\) with the Lebesgue measure \(\lambda\) and \(k = (k_n)\), \(1 = k_1 < k_2 < \ldots\).

Let us remark as in the case of constants in Section 3 that the condition

\[\sum_{l=1}^{\infty} r_j = 0\]

is stronger than (5) and therefore implies the \(\mu \times P\)-almost sure convergence of \(\sum_{l=1}^{\infty} \xi_{\pi(l)}\). The condition was stated in [17] as an existence condition. Although not necessary, (6) proves to be convenient especially when the sequence \(k = (k_n)\) is not known. Another benefit of (6) is that for the classes of Banach spaces, in contrast to (5), it can be expressed effectively in terms of individual summands \(\xi_i\)-s (see Section 3). We don't give here all the corollaries, instead we restrict ourselves with the case of scalar \(\xi_i\)-s that leads in the existence setting to the famous Nikishin and Garsia theorems.

Corollary. (a) (Nikishin, [2,]) Assume a series \(\sum_{l=1}^{\infty} \xi_l\) of real or complex random variables converges in measure and \(\sum_{l=1}^{\infty} |\xi_l| < \infty\) a.s. Then there exists a permutation \(\pi : N \to N\) such that \(\sum_{l=1}^{\infty} \xi_{\pi(l)}\) converges a.s.

(b) (Garsia, [1,]) Let \((\varphi_l) \subset L_2(T, \Sigma, \nu)\) be an orthonormal system and \((\alpha_l)\) be real or complex coefficients with \(\sum_{l=1}^{\infty} |\alpha_l| < \infty\). Then there exists a permutation \(\pi : N \to N\) such that \(\sum_{l=1}^{\infty} \alpha_{\pi(l)} \varphi_{\pi(l)}\) converges a.s.

In [18] we have found a simple straightforward way of proving the Garsia inequality that leads to Corollary (b).

The method based on inequality (2) we used in this paper can be applied to different areas of probability and analysis to get the existence or massiveness of a.s. convergent rearrangements of normalized sequences. In this way we have found various formulations of the strong laws of large numbers under rearrangements [19, 20].

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REFERENCES


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