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Mathematics

Cyclic Configurations of Spherical Quadrilaterals

Grigory Giorgadze*, Giorgi Khimshiashvili**

* *Institute of Cybernetics, Tbilisi*
** *Ilia Chavchavadze State University, Andria Razmadze Mathematical Institute, Tbilisi*

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ABSTRACT. We investigate cyclic configurations and moduli spaces of spherical quadrilaterals. For nondegenerate quadrilateral linkage, we establish that cyclic configurations are critical points of the signed area function on moduli space and their number is determined by the topology of moduli space. We also find the maximal value of the signed area on moduli space. *© 2009 Bull. Georg. Natl. Acad. Sci.*

Key words: *spherical linkage, moduli space, linkage mapping.*

1. We deal with spherical linkages defined as follows. Let S be a unit sphere in three-dimensional Euclidean space $\boldsymbol{R}^3$ and d be the distance function defined by the induced Riemannian metric on S [1]. Given a natural number k, a *spherical k-linkage L* is defined by a k-tuple of nonnegative real numbers $l_i<\pi$ (called *sidelengths* of L) each of which is not greater than the sum of all other ones [2]. The configuration space $C_S(L)$ of spherical linkage L is defined as the totality of all k-tuples of points $v_i\in S$ such that $d(v_i,v_{i+1})=l_i$. Each such collection of points is called a *configuration* of L. Factoring this set over the natural diagonal action of $SO(3)$ one obtains the *moduli space* $M_S(L)$ of spherical linkage L [2]. As in the planar case, moduli spaces are endowed with natural topologies induced by distance making them into compact topological spaces [2].

Applying a homothety with center at the origin and coefficient R we obtain a spherical linkage $R*L$ with sidelengths Rl_i on sphere S_R of radius R and can define its moduli space in S_R. It is of course obvious that the moduli spaces of L and $R*L$ are homeomorphic. Notice also that if we multiply all sidelengths by the same number $r<1$, the corresponding linkage rL has the same moduli space in S as L itself. These two observations will enable us to describe the topological type of moduli spaces of certain spherical quadrilaterals (see below Theorem 1), which is needed for formulating the main result of this note. Notice that moduli spaces of planar quadrilaterals were thoroughly studied in [3] (cf. also [4, 5]).

By analogy with the case of planar polygons, under a *cyclic spherical polygon* we understand a polygon which can be inscribed in a circle lying on S, i.e., there exists a point (center of circumscribed circle) equidistant from all vertices of the polygon (see, e.g., [6]). Study of cyclic planar polygons has a long history starting with elementary classical results such as Ptolemy theorem and Brahmagupta formula (see, e.g., [6]). Important results on existence and geometry of cyclic planar polygons were obtained by J.Steiner [6]. The study of cyclic planar polygons continues to attract considerable interest, in particular, due to the results and conjectures of D.Robbins concerned with computation of areas of cyclic planar polygons [7]. A close relation between cyclic configurations and topology of the moduli space of planar polygonal linkage was established in [8]. For planar quadrilaterals, the general constructions and conjectures of [8] were elaborated and developed in [9]. The aim of this note is to extend some results from [8] and [9] to the case of spherical linkages introduced above. Our main result (Theorem 3) states that the cyclic configurations are critical points of the signed area function for a spherical quadrilateral linkage L as above and their amount is determined by the topology of moduli space $M_S(L)$.

2. Let us begin with a few general remarks on moduli spaces of spherical linkages. Notice first that, by complete analogy with the planar case, the moduli space of a linkage L as above can be identified with the subset of configurations such that $v_1 = (1, 0, 0)$, $v_2 = (\cos l_1, \sin l_1, 0)$. Assuming that this is always the case it is easy to realize that $M_S(L)$ can be represented as a level set of a certain proper smooth mapping between affine spaces of appropriate dimensions which is called the *linkage mapping* (cf. [4]). By a standard application of Ehresmann fibration theorem we conclude that, for generic values of l_i, the moduli space $M_S(L)$ has a natural structure of compact orientable manifold of dimension $k - 3$. As was shown in [2], the condition of genericity appearing in the last statement can be made quite precise. Following [2] let us say that linkage is *degenerate* if it has a configuration all vertices of which lie on the same great circle of S. A minute thought shows that this happens if and only if there exists a k-tuple of "signs" $s_i = \pm 1$ such that $\Sigma s_i l_i = 0$.

Proposition 1. ([2]) The moduli space $M_S(L)$ is a smooth manifold if and only if linkage L is not degenerate in the above sense.

This was proven in [2] in the framework of general theory of linkages. In the case of a spherical quadrilateral which we only consider in the sequel, the linkage mapping can be written explicitly in the following form:

$$\mathbf{R}^6 = \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^5 = \mathbf{R}^3 \times \mathbf{R}^2,$$

$$(x,y,z,u,v,w) \to (\arccos (x\cos(l_i)+y\sin(l_i)),\ \arccos (xu + yv + wz),\ \arccos (u),\ x^2 + y^2 + z^2 = 1,\ u^2 + v^2 + w^2 = 1).$$

It is obvious that, for generic sidelengths, such a moduli space is a smooth compact one-dimensional manifold, i.e. it has a finite number of connected components homeomorphic to circle S^1. For a smooth moduli space, a natural idea is to investigate its topology using Morse theory of some natural smooth function on it. According to [8], in the planar case an appropriate function is given by the signed area A [6]. Obviously, the signed area is also defined for any spherical configuration as above and we denote by A_S the corresponding function on moduli space $M_s(L)$.

As was shown in [8], cyclic configurations can be interpreted as critical points of function A_S. In order to obtain similar results in our setting it is necessary to obtain first results on the moduli spaces and cyclic configurations of spherical quadrilaterals. At present we only have complete results for, so to say, "not too big" linkages. Namely, following [2] we say that a linkage on certain sphere S_R is *moderate* if its perimeter is strictly less than the length of the great circle. Obviously, this condition is preserved by homotheties. Notice that all configurations of a moderate linkage in S with the two vertices fixed as above, belong to the same hemisphere of S. Thus the stereographic projection Π of S on the tangent plane at point $v_1=(1,0,0)$ defines a one-to-one mapping on $M_S(L)$. Since for big R the distortion of Π is small compared with sidelengths, it appears possible to relate moduli spaces of moderate linkages to those of planar ones.

Proposition 2. The moduli space $M_S(L)$ of a moderate spherical linkage is homeomorphic to the moduli space of planar linkage L^* with the same sidelengths [2].

This statement is presented in [2] without proof. In the framework of our approach the proof can be obtained as follows. First of all, our interpretation of moduli spaces as the fibres of quadratic mapping enables one to show that they continuously depend on the sidelengths vector with respect to the Hausdorff distance. Next, we apply a homothety with coefficient $R > 1$ and consider linkage R^*L in the sphere of radius R, which as we know does not change either the topological type of moduli space or the moderacy of linkage. Then we shrink all sidelengths R times, which also does not change the topological type of moduli space and of course does not violate the moderacy condition (since $R > 1$). Thus we conclude that a moderate linkage can be embedded in the sphere of arbitrary big radius R without changing the topological type of its moduli space. Notice now that, for sufficiently big R, the moduli space $M_S(R)$ is close to the moduli space of the planar linkage with the same sidelengths. Since the linkage is nondegenerate we can apply the Ehresmann theorem and conclude that the two moduli spaces are in fact homeomorphic as was claimed.

This result combined with the results of [9] enables us to derive a complete topological description of moduli spaces of moderate quadrilaterals.

Theorem 1. *The moduli space of nondegenerate moderate spherical quadrilateral can be homeomorphic either to circle or to disjoint union of two circles. The number of components is one, if the sum of the longest and shortest sides is bigger than the sum of to other sides. In the opposite case the number of components is two.*

Applying the same reduction to the planar case we also obtain detailed information on cyclic configurations. Notice that the notion of convexity is naturally defined for each subset of S with diameter not exceeding 2π, in particular, for each configuration of a moderate linkage.

Theorem 2. *Each nondegenerate moderate spherical quadrilateral has a convex cyclic configuration. The number of cyclic configurations is two if the sum of the longest and shortest sides is bigger than the sum of two other sides. In the opposite case the number of cyclic configurations is four.*

Corollary 1. *The number of cyclic configurations is two times the number of components of the moduli space.*

3. It is easy to see that A_S is a differentiable function on moduli space of nondegenerate moderate spherical quadrilateral Q. Thus we can consider its critical points and it turns out that they are always nondegenerate in the sense of Morse theory.

Proposition 3. The signed area A_S is a Morse function on the moduli space of nondegenerate quadrilateral spherical linkage L.

The proof is derived from the analogous statement for planar linkages using the stereographic projection Π described above. Under our conditions Π is a diffeomorphism so the Hessians of A_S and of the area function on the moduli space of the corresponding planar linkage appear conjugated by the differential $d\Pi$. The statement follows since in the planar case the Hessian of area is nondegenerate according to [8].

After these preparations we are able to formulate the main result of this note.

Theorem 3. *Let Q be a nondegenerate moderate spherical quadrilateral linkage. Then all critical points of the signed area function A_S on moduli space $M_S(Q)$ are given by the cyclic configurations of Q. In particular, A_S attains its maximum at the convex cyclic configuration.*

Since our assumptions imply that the moduli space is one-dimensional and A_S is a Morse function, its critical points can be either maxima or minima. Consider first the configuration V at which A_S attains its maximum on $M_S(Q)$. Then using a natural analog of Steiner's classical four-linkage method one can show that V is cyclic. It follows that the global minimum of A_S is attained at configuration V^* which is obtained from V by the geodesic reflection in the first side of Q (by our agreement the latter lies on the equator of S). For local extrema the proof is more complicated and involves analysis of infinitesimal displacements in the tangent space to $M_S(Q)$.

In our opinion, this result reveals curious aspects of spherical linkages and suggests interesting problems some of which are mentioned in the sequel. In particular, it is now easy to relate the number of cyclic configurations to the topology of moduli space.

Corollary 2. *The number of critical points of A_S in $M_S(Q)$ is two times the number of components of moduli space $M_S(Q)$.*

In topological terms these results mean that A_S is a perfect Morse function on $M_S(Q)$ (cf. [8]). It seems remarkable that the maximum Σ of A_S on $M_S(L)$ can be explicitly computed as follows. Let a, b, c, d be the sidelengths of Q and p be its half-perimeter.

Proposition 4. One has:

$$tan\left(\frac{\Sigma}{4}\right)=\sqrt{tan((p-a)/2)\,tan((p-b)/2)\,tan((p-c)/2)\,tan((p-d)/2)}$$

where *tan* stays for the usual tangent function.

The proof of the proposition is obtained by decomposing the convex cyclic configuration into a union of two triangles and then applying the $S = \alpha+\beta+\gamma-\pi$ formula for the area [1] and well-known formulae for the angles of spherical triangle in terms of scalar products of radius-vectors of its vertices (all necessary results of spherical trigonometry can be found in Ch.18 of [1]). This formula can be considered as a natural analog of the classical Brahmagupta formula for the area of cyclic planar quadrilateral [6].

There also exists a similar formula for the value of a local minimum of A_S but we were not able to bring it to a compact form like the one above. Thus we obtain rather complete information on critical points of A_S on moduli spaces of moderate spherical quadrilaterals.

It is now quite natural to wonder if similar results hold for spherical k-linkages with arbitrary k (Problem 1) and what happens for linkages with perimeter bigger than the length of the great circle (Problem 2).

We do not have substantial results in either of these directions and conclude with a few related remarks. First of all, the nice relation between the topology on moduli space and number of cyclic configurations is not preserved for $k > 4$. Already for spherical pentagons it is impossible to determine the number of cyclic configurations from the topology of $M_S(P)$ because there exist pentagons P, P' such that $M_S(P)$ is homeomorphic to $M_S(P')$ but P and P' have a different number of cyclic configurations. Examples of such pentagons were found by the first-named author using computer experiments based on the second-named author's algorithm for calculation of the local degree. This implies, in particular, that A is not always a perfect Morse function on moduli spaces of pentagon linkages and we are led to the problem of characterizing those sidelengths for which A is a perfect Morse function on $M(P)$ (Problem 3). We believe that each of these three problems is quite promising and intend to continue investigation of spherical linkages within the setting outlined in this note.

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მათემატიკა

სფერული ოთხკუთხედების ციკლური კონფიგურაციები

გ. გიორგაძე*, გ. ხიმშიაშვილი**

* *კიბერნეტიკის ინსტიტუტი, თბილისი*
** *ილია ჭავჭავაძის სახელმწიფო უნივერსიტეტი და ანდრია რაზმაძის მათემატიკის ინსტიტუტი, თბილისი*

(წარმოდგენილია აკადემიკოს გ. ხარატიშვილის მიერ)

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