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Mathematics

On General Solutions of First-Order Nonlinear Matrix and Scalar Ordinary Differential Equations

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ABSTRACT. This article contains the formulae of general solutions for particular classes of first-order nonlinear matrix and scalar ordinary differential equations. *© 2009 Bull. Georg. Natl. Acad. Sci.*

Key words: *general solutions, first order matrix, scalar, nonlinear matrix and scalar ordinary differential equations.*

1. First-Order Canonical Nonlinear Equation. Problem Statement

Let us consider an equation

$$(p-A)h(t,X)=F, t\in I=]t_1,t_2[\subset]-\infty,\infty[,\ p=\frac{d}{dt}, \tag{1}$$

where $A=A(t)$, $F=F(t)$, $t\in I$ are given n x n matrices with continuous elements on the interval I, $h=h(t,X)$, $t\in I$ is a given arbitrary admissible n x n matrix function, X is an unknown n x n matrix.

Here and everywhere an admissible function will be called any function in respect to which operations presented in the article are valid on the whole interval I.

Definition 1. The solution of equation (1) will be called matrix function $X=X(t)$ defined on the interval I, substitution of which in equation (1) is admissible as a result of which we get the identity.

Definition 2. Let t_0 be an arbitrary fixed point of the interval I and X_0 be arbitrary fixed constant of matrix n x n. Matrix function $X(t, C_0)$ defined on the interval I and depending on arbitrary constant C_0 of matrix n x n will be called the general solution of equation (1), if $X(t, C_0)$, $t\in I$ is a solution of equation (1) satisfying the initial condition $X(t_0, C_0)=X_0$.

The basic problem consists in constructing the general solution of equation (1).

2. Regular Matrices. Main Theorems

To construct the general solution of equation (1) we shall need a matrix function of regular matrix.

Definition 3. Matrix $R=R(t)$ with continuous elements $r_j^i(t), i,j=\overline{1,n}$, $t\in I$ will be called a regular matrix if there exists n x n matrix function $\Phi(\int Rdt)$ definite, continuous and continuously differentiable with respect to t on the interval I, satisfying the conditions:

$$p\Phi\left(\int Rdt\right)=R\Phi\left(\int Rdt\right);\ \exists\Phi^{-1}\left(\int Rdt\right),$$
$$\Phi^{-1}\left(\int Rdt\right)\Phi\left(\int Rdt\right)=\Phi\left(\int Rdt\right)\Phi^{-1}\left(\int Rdt\right)=\mathrm{E},\ t\in I,$$

where E is a unit matrix.

Theorem 1 (Basic Theorem). *If the matrix A=A(t), $t \in I$, is regular and the admissible matrix function X= X(t, C_0), $t \in I$ satisfies the condition*

$$h(t,X) = \Phi\left(\int Adt\right)\left(C_0 + \int \Phi^{-1}\left(\int Adt\right)Fdt\right),\ t \in I,$$

then X(t, C_0), $t \in I$ is a solution of equation (1).

Proof. We have

$$ph(t,X) = A\Phi\left(\int Adt\right)\left(C_0 + \int \Phi^{-1}\left(\int Adt\right)Fdt\right) + F,\ t \in I,$$

i.e.

$$Ph(t,X) - Ah(t,X) = F,\ \forall t \in I.$$

Theorem 1 is proved □

Let $h(t,X)=g(t)q(X)$, $t \in I$, $\forall X$, where $g=g(t)$, $q=q(X)$ are arbitrary admissible functions.

From Theorem 1 it follows

Theorem 2. *Let the matrix A=A(t), $t \in I$ is regular, g=g(t), $t \in I$, q=q(τ), $\forall \tau$ are arbitrary admissible scalar functions and $\exists q^{-1}$, i.e.*

$$q^{-1}q(\tau) = qq^{-1}(\tau) = \tau,\ \forall \tau.$$

If there exists the admissible matrix function

$$X = q^{-1}\left[\frac{1}{g}\Phi\left(\int Adt\right)\left(C_0 + \int \Phi^{-1}\left(\int Adt\right)Fdt\right)\right],\ t \in I, \tag{2}$$

then matrix function (2) is the general solution of the equation

$$(p - A)g(t)q(X) = F,\ t \in I. \tag{3}$$

Proof. From formula (2) it follows

$$g(t)q(X) = \Phi\left(\int Adt\right)\left(C_0 + \int \Phi^{-1}\left(\int Adt\right)Fdt\right),\ t \in I.$$

Consequently (see Theorem 1) matrix function (2) is a solution of equation (3).

Let t_0 be an arbitrary fixed point of the interval I and X_0 be an arbitrary fixed constant of matrix *n* x *n.* Assume that $X(t_0)=X_0$. Then from formula (2) it follows

$$C_0 = g(t_0)\Phi^{-1}\left(\int Adt\right)\Big|_{t=t_0} q(X_0) - \int \Phi^{-1}\left(\int Adt\right)Fdt\Big|_{t=t}\ .$$

Theorem 2 is proved □

From Theorem 1 it follows

Theorem 3. *Let the matrix A=A(t), $t \in I$ is regular, G=G(t), $t \in I$, q=q(τ), $\forall \tau$ are arbitrary admissible functions and $\exists q^{-1}$, i.e. $q^{-1}q(\tau) = qq^{-1}(\tau) = \tau,\ \forall \tau$.*

If there exists the admissible matrix function

$$X = q^{-1}\left[\Phi\left(\int Adt\right)\left(C_0 + \int \Phi^{-1}\left(\int Adt\right)Fdt\right) - G\right],\ t \in I, \tag{4}$$

then matrix function (4) is the general solution of the equation

$$(p - A)[G + q(X)] = F,\ t \in I. \tag{5}$$

Proof. From formula (4) it follows

$$G + q(X) = \Phi\left(\int Adt\right)\left(C_0 + \int \Phi^{-1}\left(\int Adt\right)Fdt\right),\ t \in I.$$

Consequently (see Theorem 1) matrix function (4) is a solution of the equation (5).

Let t_0 be an arbitrary fixed point of the interval I and X_0 be an arbitrary fixed constant of matrix *n* x *n.* Assume that $X(t_0)=X_0$. Then from formula (4) it follows

$$C_0 = \Phi^{-1}\left(\int Adt\right)\Big|_{t=t_0} q(X_0) - \left[\int \Phi^{-1}\left(\int Adt\right)Fdt - G\right]\Big|_{t=t_0}.$$

Theorem 3 is proved.

3. Criteria of Regularity

Theorem 4 (**Criterion of Regularity**). *Let* $G = G(t) = \left(g_j^i(t)\right), t \in I$, *be an arbitrary n x n matrix with continuous and continuously differentiable elements* $g_j^i(t), i, j = \overline{1,n}$, *on the interval I. Let* $P(G) = a_0(t)E + a_1(t)G(t) + \cdots + a_k(t)G^k(t),\ t \in I$, *where* $a_m(t), m = \overline{0,k}, t \in I$, *are arbitrary continuous and continuously differentiable scalar functions.*

Let $\det P(G) \neq 0, \forall t \in I$. *Let* $R = \dot{P}(G)P^{-1}(G), t \in I,$ *and* $\exists \Phi\left(\int R dt\right) = P(G),\ t \in I$. *Then matrix R,* $t \in I$ *is regular.*

Here the dot stands for the derivative *d/dt*.

Proof. We have

$$\exists \Phi^{-1}\left(\int R dt\right) = P^{-1}(G),\ t \in I, \text{ and } \dot{\Phi}\left(\int R dt\right) = \dot{P}(G) = RP(G) = R\Phi\left(\int R dt\right),\ t \in I .$$

Theorem 4 is proved. □

Corollary 1 (Criterion of Regularity). *Let* $k = 1,\quad a_0(t) \equiv 0,\ a_1(t) \equiv 1,\ t \in I,\ \det G(t) \neq 0,\ t \in I,$ $R = \dot{G}(t)G^{-1}(t),\ t \in I,$ *and* $\exists \Phi\left(\int R dt\right) = G(t),\ t \in I.$ *Then matrix* ***R***, $t \in I$ *is regular.*

4. Applications

4.1. General Solutions of First Order Nonlinear Scalar Differential Equations

Let us consider the equation

$$(p-a)\, g(t)\, q(x) = f,\quad t \in I, \tag{6}$$

where $a=a(t), f=f(t), g=g(t),\ t \in I,\ q=q(x)$ are given arbitrary admissible scalar functions.

If n=1, from Theorem 2 it follows

Result 1. If $a=a(t), f=f(t), g=g(t),\ t \in I,\ q=q(x)$ *are arbitrary admissible scalar functions and* $\exists q^{-1}$, *i.e.* $q^{-1}q(x) = qq^{-1}(x) = x$, *then the general solution of the equation*

$$g\frac{\partial q(x)}{\partial x}\dot{x} + (\dot{g} - ag)q(x) = f,\quad t \in I, \tag{7}$$

has the form

$$x = q^{-1}\left[\frac{1}{g}e^{\int adt}\left(c_0 + \int e^{-\int adt} f dt\right)\right],\quad t \in I. \tag{8}$$

Remark. In this case $\Phi\left(\int adt\right) = e^{\int adt},\quad t \in I.$

For example, the general solution of the equation

$$g\dot{x} + fe^x + (ag - \dot{g}) = 0,\quad t \in I,$$

where $a=a(t), f=f(t), g=g(t),\ t \in I$ are arbitrary admissible functions, has the form

$$x = -\ln\left[\frac{1}{g}e^{\int adt}\left(c_0 + \int e^{-\int adt} f dt\right)\right],\ t \in I.$$

Remark. In this case $q(x)=e^{-x},\ \forall x$.

If $g \equiv 1,\ t \in I$, the general solution of the equation

$$\dot{x} + f(t)e^x + a(t) = 0,\ t \in I,$$

has the form

$$x = -\int a(t)dt - \ln\left(c_0 + \int e^{-\int a(t)dt} f(t)dt\right),\ t \in I.$$

If $n = 1$, from Theorem 3 it follows

Result 2. If a=$a(t)$, f=$f(t)$, g=$g(t)$, $t \in I$, q=$q(x)$ are arbitrary admissible scalar functions and $\exists q^{-1}$, i.e. $q^{-1}q(x) = qq^{-1}(x) = x$, then the general solution of the equation

$$\frac{\partial q}{\partial x}\dot{x} - aq(x) - (f + ag - \dot{g}) = 0, \quad t \in I, \tag{9}$$

has the form

$$x = q^{-1}\left[e^{\int adt}c_0 - g + \int e^{-\int adt} fdt\right], \quad t \in I. \tag{10}$$

For example, the general solution of the equation

$$\dot{x} - atgx - (f + ag - \dot{g})\sec x = 0, \; t \in I,$$

where a=$a(t)$, f=$f(t)$, g=$g(t)$, $t \in I$ are arbitrary admissible functions, has the form

$$x = \arcsin\left[e^{\int adt}\left(c_0 + \int e^{-\int adt} fdt\right) - g\right], \quad t \in I.$$

Remark 2. In this case $q(x)$=$\sin(x)$.

5. Nonlinear Matrix Differential Equations (Examples)

Example 1. Let $G = \begin{pmatrix} \alpha, \beta \\ 0, \; \alpha \end{pmatrix}$, $t \in I$, where $\alpha = \alpha(t), \beta = \beta(t), t \in I$, are arbitrary admissible functions. Let $\alpha \neq 0, \; \forall t \in I$ and $A = \dot{G}\,G^{-1}, \; t \in I.$ i.e. $A = \begin{pmatrix} \dfrac{\dot{\alpha}}{\alpha}, & \left(\dfrac{\beta}{\alpha}\right)^{\bullet} \\ 0, & \dfrac{\dot{\alpha}}{\alpha} \end{pmatrix}$, $t \in I$. From Corollary 1 it follows that matrix $A, \; t \in I$, is regular if $\exists \Phi(\int Adt) = G, \, t \in I \Rightarrow \Phi\begin{pmatrix} \ln\alpha, & \dfrac{\beta}{\alpha} \\ 0, & \ln\alpha \end{pmatrix} = \begin{pmatrix} \alpha, \beta \\ 0, \alpha \end{pmatrix}, \; t \in I.$

Let us consider the matrix function $e^{\int Adt} = e^{\begin{pmatrix} \ln\alpha, \frac{\beta}{\alpha} \\ 0, \; \ln\alpha \end{pmatrix}}, \; t \in I.$

We have (see [1])

$$e^{\int Adt} = \begin{pmatrix} \alpha, \beta \\ 0, \alpha \end{pmatrix}, \; t \in I.$$

Hence, $\exists \Phi\left(\int Adt\right) = e^{\int Adt} = G, \, t \in I \Rightarrow \exists \Phi^{-1}\left(\int Adt\right) = G^{-1}, \; t \in I$, and matrix A, $t \in I$, is regular.

Let us consider the equation (3), where $A = \begin{pmatrix} \dfrac{\dot{\alpha}}{\alpha}, & \left(\dfrac{\beta}{\alpha}\right)^{\bullet} \\ 0, & \dfrac{\dot{\alpha}}{\alpha} \end{pmatrix}$, $g, \; q$ are arbitrary admissible functions.

From Theorem 2 it follows

$$X = q^{-1}\left[\frac{1}{g}\begin{pmatrix} \alpha, \beta \\ 0, \alpha \end{pmatrix}\left(c_0 + \int \frac{1}{\alpha^2}\begin{pmatrix} \alpha, -\beta \\ 0, \alpha \end{pmatrix} Fdt\right)\right], \quad t \in I.$$

For example, if $q(X) = X^{\mu}, \; g(t) = \dfrac{1}{t}, \; F \equiv 0, \; C_0 = E, \; \mu - \forall const \neq 0, \; t \neq 0,$ we have

$$X=\left[t\begin{pmatrix}\alpha,\beta\\0,\alpha\end{pmatrix}\right]^{\frac{1}{\mu}},\quad t\in I,\ \text{i.e.(see [1])}\quad X=(\alpha\, t)^{\frac{1}{\mu}}\begin{pmatrix}1,\dfrac{\beta}{\mu\alpha}\\0,\ 1\end{pmatrix},\quad X^{\mu}=t\begin{pmatrix}\alpha,\beta\\0,\alpha\end{pmatrix},\ t\in I.$$

From the equation (3) it follows

$$\left[P-\begin{pmatrix}\dfrac{\dot\alpha}{\alpha}, & \left(\dfrac{\beta}{\alpha}\right)^{\bullet}\\ 0, & \dfrac{\dot\alpha}{\alpha}\end{pmatrix}\right]\frac{1}{t}X^{\mu}=\begin{pmatrix}\dot\alpha,\dot\beta\\0,\dot\alpha\end{pmatrix}-\begin{pmatrix}\dfrac{\dot\alpha}{\alpha}, & \left(\dfrac{\beta}{\alpha}\right)^{\bullet}\\ 0, & \dfrac{\dot\alpha}{\alpha}\end{pmatrix}\begin{pmatrix}\alpha,\beta\\0,\alpha\end{pmatrix}=0,\ t\in I.$$

Consequently, the particular solution $(C_0=E)$ of the equation

$$\left[p-\begin{pmatrix}\dfrac{\dot\alpha}{\alpha}, & \left(\dfrac{\beta}{\alpha}\right)^{\bullet}\\ 0, & \dfrac{\dot\alpha}{\alpha}\end{pmatrix}\right]\frac{1}{t}X^{\mu}=0,\ t\in I,$$

where $\alpha=\alpha(t),\ \beta=\beta(t),\quad t\in I$ are arbitrary admissible functions, $\mu-\forall const\neq 0$, has the form

$$X=(\alpha t)^{\frac{1}{\mu}}\begin{pmatrix}1,\dfrac{\beta}{\mu\alpha}\\0,\ 1\end{pmatrix},\quad t\in I\ \ \square$$

Example 2. Let $G=\begin{pmatrix}\alpha,\ 0\\ \beta,\alpha\end{pmatrix},\ t\in I,$ where $\alpha=\alpha(t)\neq 0,\ \forall t\in I,\ \beta=\beta(t),\ t\in I,$ are arbitrary admissible functions. Let $A=\dot G G^{-1},\ t\in I,$ i.e. $A=\begin{pmatrix}\dfrac{\dot\alpha}{\alpha}, & 0\\ \left(\dfrac{\beta}{\alpha}\right)^{\bullet}, & \dfrac{\dot\alpha}{\alpha}\end{pmatrix},\ t\in I.$

Hence, matrix A is regular if $\exists\Phi\left(\int Adt\right)=G,\ t\in I.$ Let us consider the matrix function $e^{\int Adt},\ t\in I.$ We have (see [1])

$$e^{\int Adt}=\begin{pmatrix}\alpha,0\\ \beta,\alpha\end{pmatrix}=G,\ \ t\in I,\ i.e.,\ \exists\Phi\int Adt=G,\ \ t\in I,\ \text{and matrix } A,\ t\in I,\ \text{is regular.}$$

Let us consider equation (5), where $A=\begin{pmatrix}\dfrac{\dot\alpha}{\alpha}, & 0\\ \left(\dfrac{\beta}{\alpha}\right)^{\bullet}, & \dfrac{\dot\alpha}{\alpha}\end{pmatrix},\ t\in I,\ \ q(X)=e^{X},\ \ G=g(t)E,\ g=g(t),\ t\in I,$ is an arbitrary admissible scalar function.

If $\exists q^{-1}(X)=\ln X,$ then from Theorem (3) it follows that the general solution of equation (5) has the form

$$X=\ln\left[e^{\int Adt}\left(C_0+\int e^{-\int Adt}Fdt\right)-g(t)E\right],\quad t\in I,$$

i.e.

$$X=\ln\left[\begin{pmatrix}\alpha,0\\ \beta,\alpha\end{pmatrix}\left(C_0+\int\frac{1}{\alpha^2}\begin{pmatrix}\alpha,\ 0\\ -\beta,\alpha\end{pmatrix}Fdt\right)-g(t)E\right],\ t\in I.$$

For example, let $C_0=E,\ \ F=\begin{pmatrix}\alpha,0\\ \beta,\alpha\end{pmatrix},\ t\in I.$

We have

$$X = \ln\left[\begin{pmatrix} (1+t)\alpha - g, & 0 \\ (1+t)\beta, & (1+t)\alpha - g \end{pmatrix}\right], \quad t \in I.$$

Consequently (see [1]), if $(1+t)\alpha - g > 0, \quad \forall t \in I,$

$$X = \begin{pmatrix} \ln[(1+t)\alpha - g], & 0 \\ \dfrac{(1+t)\beta}{(1+t)\alpha - g}, & \ln[(1+t)\alpha - g] \end{pmatrix}, \quad t \in I, \tag{11}$$

$$e^X = \begin{pmatrix} (1+t)\alpha - g, & 0 \\ (1+t)\beta, & (1+t)\alpha - g \end{pmatrix}, \quad t \in I.$$

Hence, the particular solution $(C_0 = E)$ of the equation

$$\left[p - \begin{pmatrix} \dfrac{\dot\alpha}{\alpha}, & 0 \\ \left(\dfrac{\beta}{\alpha}\right)^{\bullet}, & \dfrac{\dot\alpha}{\alpha} \end{pmatrix}\right]\left[\begin{pmatrix} g, 0 \\ 0, g \end{pmatrix} + e^X\right] = \begin{pmatrix} \alpha, 0 \\ \beta, \alpha \end{pmatrix}, \ t \in I,$$

where $\alpha = \alpha(t) \neq 0,\ \beta = \beta(t), g = g(t) < (1+t)\alpha,\ t \in I$ are arbitrary admissible functions, has the form (11).

In conclusion we must notice that the related problems are investigated in [2-4].

მათემატიკა

პირველი რიგის არაწრფივი სკალარული და მატრიცული ჩვეულებრივი დიფერენციალური განტოლებების ზოგადი ამონახსნების შესახებ

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