On General Solutions of First-Order Nonlinear Matrix and Scalar Ordinary Differential Equations

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1. First-Order Canonical Nonlinear Equation. Problem Statement

Let us consider an equation

\[(p-A)h(t, X) = F, t \in I = \mathbb{I}_1, t_2 [\infty, \omega] \setminus \mathbb{I}_1, P = \frac{d}{dt},\]

where \(A = A(t), F = F(t), t \in I\) are given \(n \times n\) matrices with continuous elements on the interval \(I\), \(h = h(t, X), t \in I\) is a given arbitrary admissible \(n \times n\) matrix function, \(X\) is an unknown \(n \times n\) matrix.

Definition 1. The solution of equation (1) will be called matrix function \(X = X(t)\) defined on the interval \(I\), substitution of which in equation (1) is admissible as a result of which we get the identity.

Definition 2. Let \(t_0\) be an arbitrary fixed point of the interval \(I\) and \(X_0\) be arbitrary fixed constant of matrix \(n \times n\). Matrix function \(X(t, C_0)\) defined on the interval \(I\) and depending on arbitrary constant \(C_0\) of matrix \(n \times n\) will be called the general solution of equation (1), if \(X(t, C_0), t \in I\) is a solution of equation (1) satisfying the initial condition \(X(t_0, C_0) = X_0\).

The basic problem consists in constructing the general solution of equation (1).

2. Regular Matrices. Main Theorems

To construct the general solution of equation (1) we shall need a matrix function of regular matrix.

Definition 3. Matrix \(R = R(t)\) with continuous elements \(r_{ij}(t), i, j = 1, n, t \in I\) will be called a regular matrix if there exists \(n \times n\) matrix function \(\Phi(\int R dt)\) definite, continuous and continuously differentiable with respect to \(t\) on the interval \(I\), satisfying the conditions:

\[p \Phi(\int R dt) = R \Phi(\int R dt); \exists \Phi^{-1}(\int R dt),\]

\[\Phi^{-1}(\int R dt) = \Phi(\int R dt) \Phi^{-1}(\int R dt) = E, t \in I,\]

where \(E\) is a unit matrix.
**Theorem 1 (Basic Theorem).** If the matrix $A=A(t)$, $t \in I$, is regular and the admissible matrix function $X=X(t, C_0)$, $t \in I$ satisfies the condition
\[
h(t, X) = \Phi \left( \int A(t) dt \left| C_0 + \int \Phi^{-1} \left( \int A(t) F(t) dt \right) \right. \right), \quad t \in I,
\]
then $X(t, C_0)$, $t \in I$ is a solution of equation (1).

**Proof.** We have
\[
ph(t, X) = A \Phi \left( \int A(t) dt \left| C_0 + \int \Phi^{-1} \left( \int A(t) F(t) dt \right) + F, \quad t \in I,\right. \right)
\]
i.e.
\[
Ph(t, X) - Ah(t, X) = F, \quad \forall t \in I.
\]

Theorem 1 is proved $\Box$

Let $h(t, X) = g(t) q(X)$, $t \in I$, $\forall X$, where $g=g(t)$, $q=q(X)$ are arbitrary admissible functions.

From Theorem 1 it follows

**Theorem 2.** Let the matrix $A=A(t)$, $t \in I$ is regular, $g=g(t)$, $t \in I$, $q=q(t)$, $\forall t$ are arbitrary admissible scalar functions and $\exists q^{-1}$, i.e.
\[
q^{-1}(\tau) = q q^{-1}(\tau) = \tau, \quad \forall \tau.
\]

If there exists the admissible matrix function
\[
X = q^{-1} \left[ \frac{1}{g} \Phi \left( \int A(t) dt \left| C_0 + \int \Phi^{-1} \left( \int A(t) F(t) dt \right) \right. \right. \right], \quad t \in I, \quad (2)
\]
then matrix function (2) is the general solution of the equation
\[
(p - A) g(t) q(X) = F, \quad t \in I. \quad (3)
\]

**Proof.** From formula (2) it follows
\[
g(t) q(X) = \Phi \left( \int A(t) dt \left| C_0 + \int \Phi^{-1} \left( \int A(t) F(t) dt \right) \right. \right), \quad t \in I.
\]

Consequently (see Theorem 1) matrix function (2) is a solution of equation (3).

Let $t_0$ be an arbitrary fixed point of the interval $I$ and $X_0$ be an arbitrary fixed constant of matrix $n \times n$. Assume that $X(t_0) = X_0$. Then from formula (2) it follows
\[
C_0 = g(t_0) \Phi^{-1} \left( \int A(t) dt \right|_{t=t_0} q(X_0) \right. - \int \Phi^{-1} \left( \int A(t) F(t) dt \right) \right|_{t=t_0}.
\]

Theorem 2 is proved $\Box$

From Theorem 1 it follows

**Theorem 3.** Let the matrix $A=A(t)$, $t \in I$ is regular, $G=G(t)$, $t \in I$, $q=q(t)$, $\forall t$ are arbitrary admissible functions and $\exists q^{-1}$, i.e.
\[
q^{-1}(\tau) = q q^{-1}(\tau) = \tau, \quad \forall \tau.
\]

If there exists the admissible matrix function
\[
X = q^{-1} \left[ \Phi \left( \int A(t) dt \left| C_0 + \int \Phi^{-1} \left( \int A(t) F(t) dt \right) \right. \right. \right] - G, \quad t \in I, \quad (4)
\]
then matrix function (4) is the general solution of the equation
\[
(p - A) (G + q(X)) = F, \quad t \in I. \quad (5)
\]

**Proof.** From formula (4) it follows
\[
G + q(X) = \Phi \left( \int A(t) dt \left| C_0 + \int \Phi^{-1} \left( \int A(t) F(t) dt \right) \right. \right), \quad t \in I.
\]

Consequently (see Theorem 1) matrix function (4) is a solution of the equation (5).

Let $t_0$ be an arbitrary fixed point of the interval $I$ and $X_0$ be an arbitrary fixed constant of matrix $n \times n$. Assume that $X(t_0) = X_0$. Then from formula (4) it follows
\[
C_0 = \Phi^{-1} \left( \int A(t) dt \right|_{t=t_0} q(X_0) \right. - \int \Phi^{-1} \left( \int A(t) F(t) dt \right) \right|_{t=t_0} - G.
\]

Theorem 3 is proved.
3. Criteria of Regularity

**Theorem 4 (Criterion of Regularity).** Let \( G = G(t) = (g_{ij}(t)), t \in I, \) be an arbitrary \( n \times n \) matrix with continuous and continuously differentiable elements \( g_{ij}(t), i, j = 1, n, \) on the interval \( I. \) Let \( P(G) = a_{ij}(t)g_j(t) + \cdots + a_k(t)g^k(t), t \in I, \) where \( a_{ij}(t), m = 0, k, t \in I, \) are arbitrary continuous and continuously differentiable scalar functions.

Let \( \det P(G) \neq 0, \forall t \in I. \) Let \( R = \dot{P}(G)^{-1}(G), t \in I, \) and \( \exists \Phi \left( \int R dt \right) = P(G), t \in I. \) Then matrix \( R, t \in I \) is regular.

Here the dot stands for the derivative \( d/dt. \)

**Proof.** We have

\[
\exists \Phi^{-1} \left( \int R dt \right) = P^{-1}(G), t \in I, \quad \text{and} \quad \Phi \left( \int R dt \right) = \dot{P}(G) = R \Phi \left( \int R dt \right), t \in I.
\]

Theorem 4 is proved. \( \square \)

**Corollary 1 (Criterion of Regularity).** Let \( k = 1, \quad a_0(t) = 0, \quad a_1(t) = 1, \quad t \in I, \quad \det G(t) \neq 0, \quad t \in I, \quad R = \dot{G}(t)G^{-1}(t), \quad t \in I, \) and \( \exists \Phi \left( \int R dt \right) = G(t), t \in I. \) Then matrix \( R, t \in I \) is regular.

4. Applications

4.1. General Solutions of First Order Nonlinear Scalar Differential Equations

Let us consider the equation

\[
(p-a) g(t) q(x(t)) = f(t), \quad t \in I, \quad (6)
\]

where \( a = a(t), f = f(t), g = g(t), t \in I, \) \( q = q(x(t)) \) are given arbitrary admissible scalar functions.

If \( n = 1, \) from Theorem 2 it follows

**Result 1.** If \( a = a(t), f = f(t), g = g(t), t \in I, \) \( q = q(x(t)) \) are arbitrary admissible scalar functions and \( \exists q^{-1} \), i.e.

\[ q^{-1}(x) = qq^{-1}(x) = x, \]

then the general solution of the equation

\[
g \frac{dq(x)}{dx} \dot{x} + (\dot{g} - ag)q(x) = f, \quad t \in I,
\]

has the form

\[
x = q^{-1} \left[ \frac{1}{g} \int e^{\int_a^b c_0 + \int_a^b f(t)dt} \right], \quad t \in I.
\]

**Remark.** In this case \( \Phi \left( \int_a^b dt \right) = e^{\int_a^b t}, \quad t \in I. \)

For example, the general solution of the equation

\[ g\dot{x} + f(t)e^x + (ag - \dot{g}) = 0, \quad t \in I, \]

where \( a = a(t), f = f(t), g = g(t), t \in I \) are arbitrary admissible functions, has the form

\[
x = -\ln \left[ \frac{1}{g} e^{\int_a^b c_0 + \int_a^b f(t)dt} \right], \quad t \in I.
\]

**Remark.** In this case \( q(x) = e^x, \quad \forall x. \)

If \( g = 1, \quad t \in I, \) the general solution of the equation

\[ \dot{x} + f(t)e^x + a(t) = 0, \quad t \in I, \]

has the form

\[
x = -\int a(t)dt - \ln \left( c_0 + \int e^{\int_a^b f(t)dt} \right), \quad t \in I.
\]
If \( n = 1 \), from Theorem 3 it follows

**Result 2.** If \( a = a(t), f = f(t), g = g(t) \), \( t \in I \), \( q = q(x) \) are arbitrary admissible scalar functions and \( \exists q^{-1} \), i.e. \( q^{-1}(x) = q(x) = x \), then the general solution of the equation

\[
\dot{x} - ax(x) - f(x) = 0, \quad t \in I,
\]

has the form

\[
x = q^{-1}\left[ e^{\int_{a}^{b} c_0 - g + \int_{a}^{b} e^{\int_{a}^{b} f dt} \right], \quad t \in I.
\]

For example, the general solution of the equation

\[
\dot{x} - atx + f(x) - g(x) \sec x = 0, \quad t \in I,
\]

where \( a = a(t), f = f(t), g = g(t) \), \( t \in I \) are arbitrary admissible functions, has the form

\[
x = \arcsin\left[ e^{\int_{a}^{b} c_0 + \int_{a}^{b} e^{\int_{a}^{b} f dt} \right] - g], \quad t \in I.
\]

**Remark 2.** In this case \( q(x) = \sin(x) \).

## 5. Nonlinear Matrix Differential Equations (Examples)

**Example 1.** Let \( G = \begin{pmatrix} \alpha, \beta \\ 0, \alpha \end{pmatrix}, \ t \in I \), where \( \alpha = \alpha(t), \beta = \beta(t), t \in I \), are arbitrary admissible functions. Let

\[
\alpha \neq 0, \ \forall t \in I \text{ and } A = G^{-1}, \ t \in I. \text{ i.e. } A = \begin{pmatrix} \alpha, \ eta \\ \alpha, \ 0 \end{pmatrix}, \ t \in I. \text{ From Corollary 1 it follows that matrix } A, \ t \in I,
\]

is regular if \( \exists \Phi\left( \int A dt \right) = G, \ t \in I \Rightarrow \Phi\left( \ln \alpha, \ \frac{\beta}{\alpha} \\ 0, \ln \alpha \right) = \begin{pmatrix} \alpha, \beta \\ 0, \alpha \end{pmatrix}, \ t \in I.\)

Let us consider the matrix function \( e^{\int A dt} = e^{\int \begin{pmatrix} \alpha, \beta \\ 0, \alpha \end{pmatrix} dt} \), \( t \in I. \)

We have (see [1])

\[
e^{\int A dt} = \begin{pmatrix} \alpha, \beta \\ 0, \alpha \end{pmatrix}, \ t \in I.
\]

Hence, \( \exists \Phi\left( \int A dt \right) = e^{\int A dt} = G, \ t \in I \Rightarrow \exists \Phi^{-1}\left( \int A dt \right) = G^{-1}, \ t \in I, \text{ and matrix } A, \ t \in I, \text{ is regular.}\)

Let us consider the equation (3), where \( A = \begin{pmatrix} \alpha, \beta \\ 0, \alpha \end{pmatrix} \), \ g, q \ are arbitrary admissible functions.

From Theorem 2 it follows

\[
X = q^{-1}\left[ \frac{1}{g} \left( \begin{pmatrix} \alpha, \beta \\ 0, \alpha \end{pmatrix} \right) c_0 + \int_{a}^{b} \frac{1}{\alpha^2} \left( \begin{pmatrix} \alpha, \beta \\ 0, \alpha \end{pmatrix} \right) \right], \quad t \in I.
\]

For example, if \( q(X) = X^\mu, \ g(t) = \frac{1}{t}, \ F = 0, \ C_0 = E, \ \mu - \forall \text{ const } \neq 0, \ t \neq 0 \), we have
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From the equation (3) it follows

$$X = \left[ \frac{1}{\alpha \beta} \right]^\mu, \ t \in I, \ \text{i.e. (see [1])} \ X = (at)^\mu \begin{bmatrix} 1, \frac{\beta}{\mu \alpha} \\ 0, 1 \end{bmatrix}, \ X^\mu = g(t), \ t \in I.$$

Consequently, the particular solution \( C_0 = E \) of the equation

$$p - \begin{bmatrix} \frac{\hat{\alpha}}{\alpha} & \frac{\beta}{\alpha} \\ 0 & \frac{\hat{\alpha}}{\alpha} \end{bmatrix} \frac{1}{t} X^\mu = 0, \ t \in I,$$

where \( \alpha = \alpha(t), \beta = \beta(t), \ t \in I \) are arbitrary admissible functions, \( \mu - \forall \text{const} \neq 0 \), has the form

$$X = (at)^\mu \begin{bmatrix} 1, \frac{\beta}{\mu \alpha} \\ 0, 1 \end{bmatrix}, \ t \in I \ \Box$$

**Example 2.** Let \( G = \left( \frac{\alpha}{\beta}, 0 \right) \), \( t \in I \), where \( \alpha = \alpha(t) \neq 0, \forall t \in I, \beta = \beta(t), \ t \in I \), are arbitrary admissible functions. Let \( A = \hat{G}G^{-1}, \ t \in I, \ \text{i.e.} \ A = \begin{bmatrix} \frac{\hat{\alpha}}{\alpha} & 0 \\ \frac{\beta}{\alpha} & \frac{\hat{\alpha}}{\alpha} \end{bmatrix} \), \( t \in I \).

Hence, matrix \( A \) is regular if \( \exists \Phi \int A \text{dt} = G, \ t \in I \). Let us consider the matrix function \( e^{\int A \text{dt}}, \ t \in I \). We have (see [1])

$$e^{\int A \text{dt}} = \begin{bmatrix} \alpha & 0 \\ \beta \alpha \end{bmatrix} = G, \ t \in I, \ \text{i.e.}, \ \exists \Phi \int A \text{dt} = G, \ t \in I, \ \text{and matrix} \ A, \ t \in I, \ \text{is regular.}$$

Let us consider equation (5), where \( A = \begin{bmatrix} \frac{\hat{\alpha}}{\alpha} & 0 \\ \frac{\beta}{\alpha} & \frac{\hat{\alpha}}{\alpha} \end{bmatrix} \), \( t \in I \), \( q(X) = e^X, \ G = g(t)E, \ g = g(t), \ t \in I \), is an arbitrary admissible scalar function.

If \( \exists q^{-1}(X) = \ln X \), then from Theorem (3) it follows that the general solution of equation (5) has the form

$$X = \ln \left[ e^{\int A \text{dt}} \left( C_0 + \int e^{-\int A \text{dt}} F \text{dt} \right) - g(t)E \right], \ t \in I,$$

i.e.

$$X = \ln \left[ \begin{bmatrix} \alpha & 0 \\ \beta \alpha \end{bmatrix} C_0 + \frac{1}{\alpha^2} \begin{bmatrix} \alpha & 0 \\ -\beta \alpha \end{bmatrix} \int e^{-\int A \text{dt}} F \text{dt} - g(t)E \right], \ t \in I.$$
\[ X = \ln \begin{bmatrix} (1+t)\alpha - g, & 0 \\ (1+t)\beta, & (1+t)\alpha - g \end{bmatrix}, \quad t \in I. \]

Consequently (see [1]), if \( (1+t)\alpha - g > 0, \quad \forall t \in I, \)
\[ X = \begin{bmatrix} \ln[(1+t)\alpha - g], & 0 \\ (1+t)\beta, & \ln[(1+t)\alpha - g] \end{bmatrix}, \quad t \in I, \]

\[ e^x = \begin{bmatrix} (1+t)\alpha - g, & 0 \\ (1+t)\beta, & (1+t)\alpha - g \end{bmatrix}, \quad t \in I. \]

Hence, the particular solution \((C_0 = E)\) of the equation
\[
\begin{bmatrix} \alpha, & 0 \\ \beta, & \alpha \end{bmatrix} \begin{bmatrix} g, 0 \\ 0, g \end{bmatrix} = \begin{bmatrix} \alpha, 0 \\ \beta, \alpha \end{bmatrix}, \quad t \in I,
\]
where \( \alpha = \alpha(t) \neq 0, \beta = \beta(t), g = g(t) < (1+t)\alpha, \quad t \in I \) are arbitrary admissible functions, has the form (11).

In conclusion we must notice that the related problems are investigated in [2-4].

\[ \text{REFERENCES} \]