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Cybernetics

Analysis of Priority Queuing System for Replacements and Renewals

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ABSTRACT. The article considers a closed queuing system for service of replacement and renewal operations. Replacement has absolute priority regarding renewal. Replacement time and renewal time are exponential random variables. Graphic system is constructed for two-dimensional random processes, hence the linear system of algebraic equation expressing the final state of the system is written directly. With the help of the proposed graphic scheme the calculation formulas of final probabilities of system states are received in explicit form. © *2009 Bull. Georg. Natl. Acad. Sci.*

Key words: *closed queuing system, replacement, renewal, explicit solution.*

In [1] the technical system consisting of m main and n stand-by elements is considered. All elements are identical. For the system to function normally it is necessary to maintain all m main elements in operating condition. The system continues to function even if the number of main elements is reduced but the efficiency of functioning of the system drops. As a result it is necessary to apply to stand-by element and replace the failed main element with stand-by one. The intensity of failure of the main element is a, and that of stand-by one is β. The failed main element is replaced with workable stand-by element at the very first possibility. If there is no necessity in the system to make replacement (all m main elements are in operating state), renewal of the failed elements takes place. The intensities of replacement and renewal are λ and μ respectively.

In order to construct a stochastic model of the considered system we introduce the notion of system state. Say the system is in state $s(i,j)$ ($i=\overline{0,n}$, $j=\overline{0,n+m}$) if the number of missing main elements is equal to i and the number of failed elements (main and stand-by) is equal to j. Denote final probabilities through $p(i,j)$, i.e., the probability that in steady state ($t \to \infty$) the system will be in state $s(i,j)$ [1,2].

The mentioned article also gives the construction of graphical scheme – “the map of system state” that enables direct writing of the system of algebraic equations for determination of $p(i,j)$. Namely, on the “map” given in (Fig.1) each point with coordinate (i,j) corresponds to system state $s(i,j)$. The system of algebraic equations

$$(\alpha(m-i) + \beta(n-(j-i)) + v_{i,j}\lambda + u_{i,j}\,\mu)\,p(i,j) =$$
$$= \alpha(m-i+1)p(i-1,j-1)+\beta\,(n-j+i+1)p(i,j-1) +$$
$$+ \lambda p(i+1,j)+ u_{i,j+1}\,\mu\,p(i,j+1) \quad i=\overline{0,m}\,,\;\; j=\overline{i,n+i}\,. \quad (1)$$

where $v_{i,j} = 0$ if (i=0 or j=n+i) else $v_{i,j}$=1. $u_{i,j}$=1- $v_{i,j}$, but $u_{0,0}$=1; $p(i,j)$=0 if ((i<0)V(i>m)) or ((i>j)V(J>n+i)); $i=\overline{0,m}$, $j=\overline{i,n+i}$, is obtained by the following rule: the request flow bringing the system into state $s(i,j)$ equals the request flow bringing the system out of this state.

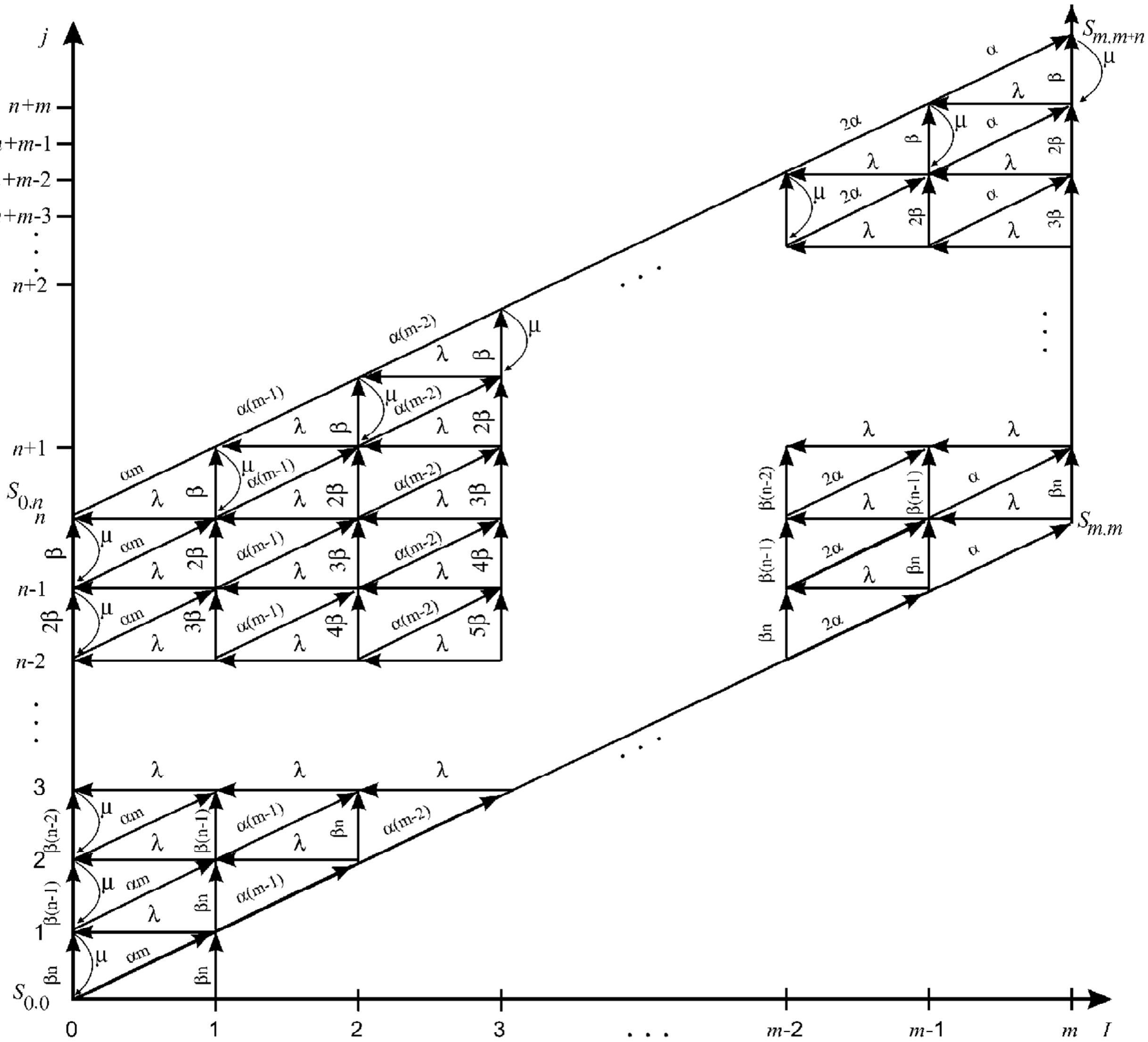

Fig. 1. The map of system states.

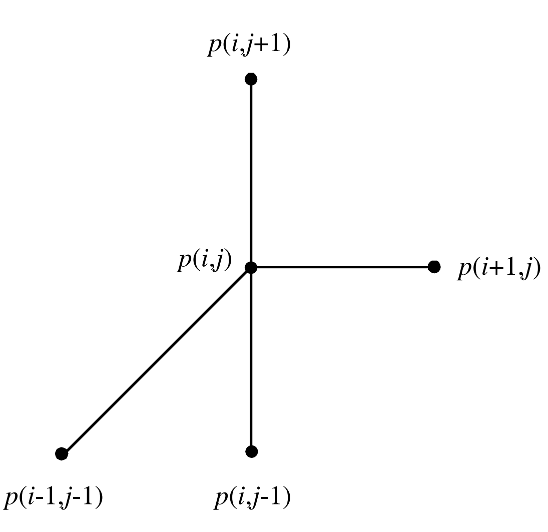

Fig. 2. The stencil of equations (1).

Note that equations in the system (1) are not difference equations in pure form. Accordingly, it is difficult to solve the system (1) by methods of the theory of difference equations.

Consider the stencil of equations (1) (Fig.2). In correspondence to the location of point $s(i,j)$ on the map (on lower or upper diagonal, on extreme verticals, etc.) the given stencil may for some $p(i,j)$ "lose" point. Taking this and also some peculiarities of our system into account, we may successively express all $p(i,j)$ ($i=\overline{1,m}$, $j=\overline{i,n+i}$) in terms of $p(0,0)$. Below, the sequence of determination of unknowns $p(i,j)$ ($i=\overline{1,m}$, $j=\overline{i,n+i}$) is given in the form of Table. In the Table the columns are written respectively: (I) – number of spacing; (II) – states of system $s(i,j)$ for which the corresponding equa-

Table

The sequence of determination of $p(i,j)$

I	II	III
1	$s(1,1)$	$p(1,1)$
2	$s(2,2)$	$p(2,2)$
…	…	…
m	$s(m,m)$	$p(m,m)$
m+1	$s(0,0)$	$p(0,1)$
m+2	$s(1,2)$	$p(1,2)$
m+3	$s(2,3)$	$p(2,3)$
…	…	…
2m+1	$s(m,m+1)$	$p(m,m+1)$
2m+2	$s(0,1)$	$p(0,2)$

I	II	III
2m+3	$s(1,3)$	$p(1,3)$
…	…	…
$(n\text{-}1)m+n\text{-}2$	$s(m,m+n\text{-}2)$	$p(m,m+n\text{-}2)$
$(n\text{-}1)m+n\text{-}1$	$s(0,n\text{-}2)$	$p(0,n\text{-}1)$
$(n\text{-}1)m+n$	$s(0,n\text{-}1)$	$p(0,n)$
$(n\text{-}1)m+n+1$	$s(0,n)$	$p(1,n)$
$(n\text{-}1)m+n+2$	$s(1,n)$	$p(1,n+1)$
…	…	…
$(n\text{-}1)m+n+2m\text{-}1$	$s(m\text{-}1,m+n\text{-}1)$	$p(m,m+n\text{-}1)$
$(n\text{-}1)m+n+2m$	$s(m,m+n\text{-}1)$	$p(m,m+n)$

tion from (1) is written down; (III) – unknowns $p(i,j)$ which are determined from this equation.

Reasoning from the algorithm shown in the Table we successively express the unknowns $p(1,1), p(2,2),\ldots, p(m,m+n)$ in terms of $p(0,0)$ by recurrent formulas of (1). Further considering the condition of normalization $\sum_{i=0}^{m}\sum_{j=i}^{n+i} p(i,j)=1$, we get formulas for determination of $p(i,j)$:

$$p(0,0) = \Big(\sum_{i=0}^{m}\sum_{j=0}^{n} q(i,j)\Big)^{-1},\quad p(i,i+j) = q(j,i)\, p(0,0),$$

$$i=\overline{0,m},\quad j=\overline{0,n}$$

where $q(0,i) = \dfrac{\alpha^i m!}{(m-i)!}\Big(\prod_{k=1}^{i}(\alpha(m-k)+\beta n+\lambda)\Big)^{-1}$,

$i=\overline{1,m}$, $q(0,0)=1$

$$q(j,i) = k_{ij}(\alpha(m\text{-}i+1)q(j,i\text{-}1) + \beta(n\text{-}j+1)q(j\text{-}1,i) +$$

$$+\lambda q(j\text{-}1,i+1)),\quad j=\overline{1,n-2},\ i=\overline{1,m},$$

where $k_{ij} = (\alpha(m\text{-}i)+\beta(n\text{-}j)+\lambda)^{-1}$, $q(j,m+1)=0$

$$q(j,0) = \mu^{-1}((\alpha m+\beta(n\text{-}j+1)+\mu)q(j\text{-}1,0)\ -$$

$$-\beta(n\text{-}j+2)q(j\text{-}2,0)\ \text{-}\lambda q(j\text{-}2,1)),\quad j=\overline{2,n}$$

$$q(1,0)=\mu^{-1}(\alpha m+\beta n)$$

$$q(n\text{-}1,i) = \lambda^{-1}((\alpha(m\text{-}i+1)+\mu)q(n,i\text{-}1)\ \text{-}$$

$$\text{-}w\alpha(m\text{-}i+2)q(n,i\text{-}2)\text{-}\ \alpha q(n\text{-}1,i\text{-}1)),\quad i=\overline{1,m}$$

where, if i=1 then w=0, else w=1.

$$q(n,i) = \mu^{-1}((\alpha(m\text{-}i)+\beta+\lambda)q(n\text{-}1,i)-$$

$$-\alpha(m\text{-}i+1)q(n\text{-}1,i\text{-}1)-2\beta q(n\text{-}2,i)\text{-}\beta q(n\text{-}2,i+1)),\quad i=\overline{1,m},$$

here $q(\text{-}1,i)=0$.

In conclusion we note that having determined the limiting probabilities of technical system states we can solve different practical problems such, for example, as the problem of renewal of technical systems or determination of their economical effectiveness.

კიბერნეტიკა

ჩანაცვლებებისა და აღდგენათა რიგების მომსახურების პრიორიტეტული სისტემის ანალიზი

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ნაშრომში განხილულია ჩანაცვლებებისა და აღდგენათა რიგების მომსახურების სისტემა ჩანაცვლების მომსახურების აბსოლუტური პრიორიტეტით. ჩანაცვლებისა და აღდგენის დრო ექსპონენციალურად განაწილებული შემთხვევითი სიდიდეებია. ორგანზომილებიანი შემთხვევითი პროცესისათვის აგებულია გრაფიკული სქემა, საიდანაც უშუალოდ ჩაწერილია სისტემის ფინალური მდგომარეობის ამსახველი წრფივ ალგებრულ განტოლებათა სისტემა. შემოთავაზებული გრაფიკული სქემის დახმარებით, ცხადი სახით, მიღებულია სისტემის მდგომარეობათა ფინალური ალბათობების გამოსათვლელი ფორმულები.

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