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On Estimation of the Remainder Term of the Generalized Sampling Series

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ABSTRACT. In the present paper estimates of the remainder term of the generalized sampling series are considered. Their comparisons for the small and large numerical values of the included $\boldsymbol{\beta}$ parameter are given.

***Key words**: entire function, random process, $L^{\alpha}(\Omega)$-process.*

The generalized sampling formula is given in [1]. The proof of this formula is analogous to the theorem 1 from [2] and is based on the following estimate: if $f(z)$ is an entire function, which satisfies the condition

$$|f(z)| \le L_f(1+|z|^m)e^{\sigma|y|}, \quad z = x+iy, \tag{1}$$

for some non-negative integer *m* and the positive real numbers L_f and σ [2], then for the fixed arbitrary z and for all sufficiently large positive integer *n* we shall have

$$\Bigg|\frac{1}{p!}\lim_{\zeta\to z}\frac{d^p}{d\zeta^p}\left(\frac{f(\zeta)}{(ae^{\delta\zeta}+be^{-\delta\zeta})\sin^{N+1}(\alpha\zeta)}\left(\frac{\sin\beta(\zeta-z)}{\beta(\zeta-z)}\right)^q\right) - \sum_{k=-n}^{n}\left(\frac{(-1)^k}{\alpha\left(z-\frac{k\pi}{\alpha}\right)}\right)^{N+1}\cdot$$

$$\left\{\sum_{\tau=0}^{N}\frac{\varphi_{\tau N}(z;k,q,\alpha,\beta,a,b,\delta)}{(N-\tau)!\,\alpha^{N-\tau}}\left[\sum_{\mu=0}^{\tau}\frac{\left[\alpha\left(z-\frac{k\pi}{\alpha}\right)\right]^{N-\mu}}{(\tau-\mu)!}\cdot\right.\right.$$

$$\left.\left.\cdot A_{\mu\tau N}\cdot\sum_{j=0}^{\mu}B_{p\mu j}\left(z-\frac{k\pi}{\alpha}\right)^j\cdot f^{(j)}\left(\frac{k\pi}{\alpha}\right)\right]\right\}\cdot\frac{1}{\left(z-\frac{k\pi}{\alpha}\right)^p}\Bigg| \le L_f\cdot C_{pqN}(z;a,b,\beta)\cdot$$

$$\cdot\frac{e^{-\delta\left(n+\frac{1}{2}\right)\frac{\pi}{\alpha}}}{\beta^q[(N+1)\alpha-\sigma-q\beta-\delta]}\cdot\left[\left(\frac{\alpha}{\pi\left(n+\frac{1}{2}\right)}\right)^{p+q+1}+\left(\frac{\alpha}{\pi\left(n+\frac{1}{2}\right)}\right)^{p+q+1-m}\right],\quad z\ne\frac{k\pi}{\alpha},\ k=0,\pm1,\pm2,\ldots. \tag{2}$$

The function $C_{pqN}(z;a,b,\beta)=\frac{2^{p+q+1}e^{q\beta|y|}}{D_0(a,b)}\left(\frac{2}{1-e^{-\pi}}\right)^{N+1}$, where $D_0(a,b)=D_0(b,a)=\min\{a,b,|a-b|\}$, is finite on the arbitrarily bounded domain of the variable z (when $\delta>0$, then we have "exponential" convergence).

Note that in the right hand side of inequality (3) from [1] (respectively in inequality (5) from [1]) the multiplier $\frac{1}{\beta^q\cdot[(N+1)\alpha-\sigma-q\sigma-\delta]}$ is omitted. The estimate (2) can be obtained if we apply Cauchy's residuals theorem for the integral $\frac{1}{2\pi i}\int_{C_n}\frac{f(\zeta)}{(ae^{\delta\zeta}+be^{-\delta\zeta})sin^{N+1}(\alpha\zeta)}\left(\frac{sin\beta(\zeta-z)}{\beta(\zeta-z)}\right)^q\frac{d\zeta}{(\zeta-z)^{p+1}}$ and estimate this integral, where C_n is circle $|\zeta|=\left(n+\frac{1}{2}\right)\frac{\pi}{\alpha}$. To obtain the estimate (2) the following inequality is used:

$$\left|\frac{\sin z}{z}\right|\leq\frac{e^{|y|}}{|z|},\qquad z=x+iy. \tag{3}$$

The inequality

$$\left|\frac{\sin z}{z}\right|\leq e^{|y|},\qquad z=x+iy \tag{4}$$

is correct also. The inequality (4) with $|z|>1$ is rougher than (3), while with $|z|<1$ inequality (4) is softer than (3). If we apply the inequality (4), then instead of (2) we shall obtain the following estimate

$$\left|\frac{1}{p!}\lim_{\zeta\to z}\frac{d^p}{d\zeta^p}\left(\frac{f(\zeta)}{(ae^{\delta\zeta}+be^{-\delta\zeta})\sin^{N+1}(\alpha\zeta)}\left(\frac{\sin\beta(\zeta-z)}{\beta(\zeta-z)}\right)^q\right)-\sum_{k=-n}^{n}\left(\frac{(-1)^k}{\alpha\left(z-\frac{k\pi}{\alpha}\right)}\right)^{N+1}\cdot\right.$$
$$\left.\cdot\left\{\sum_{\tau=0}^{N}\frac{\varphi_{\tau N}(z;k,q,\alpha,\beta,a,b,\delta)}{(N-\tau)!\,\alpha^{N-\tau}}\left[\sum_{\mu=0}^{\tau}\frac{\left[\alpha\left(z-\frac{k\pi}{\alpha}\right)\right]^{N-\mu}}{(\tau-\mu)!}\cdot A_{\mu\tau N}\cdot\left(\sum_{j=0}^{\mu}B_{p\mu j}\left(z-\frac{k\pi}{\alpha}\right)^j\cdot f^{(j)}\left(\frac{k\pi}{\alpha}\right)\right)\right]\right\}\cdot\frac{1}{\left(z-\frac{k\pi}{\alpha}\right)^p}\right|\leq$$
$$\leq L_f\cdot C_{pqN}{}^0(z;a,b,\beta)\cdot\frac{e^{-\delta\left(n+\frac{1}{2}\right)\frac{\pi}{\alpha}}}{(N+1)\alpha-\sigma-q\beta-\delta}\left[\left(\frac{\alpha}{\pi\left(n+\frac{1}{2}\right)}\right)^p+\left(\frac{\alpha}{\pi\left(n+\frac{1}{2}\right)}\right)^{p-m}\right], \tag{5}$$

where $C_{pqN}{}^0(z;a,b,\beta)=\frac{2^{p+1}e^{q\beta|y|}}{D_0(a,b)}\left(\frac{2}{1-e^{-\pi}}\right)^{N+1}$. So we see that the estimate (5) is suitable for small values of parameter β, even at $\beta=0$, however, in this case the estimate (2) makes no sense. When $\beta\to0$ then the right hand side of the inequality (2) converges to infinity, and the right hand side of the inequality (5) converges to the value

$$L_f\cdot C_{pqN}{}^0(z;a,b,0)\cdot\frac{e^{-\delta\left(n+\frac{1}{2}\right)\frac{\pi}{\alpha}}}{(N+1)\alpha-\sigma-\delta}\left[\left(\frac{\alpha}{\pi\left(n+\frac{1}{2}\right)}\right)^p+\left(\frac{\alpha}{\pi\left(n+\frac{1}{2}\right)}\right)^{p-m}\right],$$

where $C_{pqN}{}^0(z;a,b,0)=\frac{2^{p+1}}{D_0(a,b)}\left(\frac{2}{1-e^{-\pi}}\right)^{N+1}$.

It should be noted that under specific conditions formula (1) from [1] is true in a certain sense for $L^\alpha(\Omega)$ random proceses also (see [3]). By analogy with the formula (4) from [1], for n-dimensional random field we have

$$\xi(t_1,\dots,t_n)=\prod_{k=1}^{n}\left(a_ke^{\delta_kt_k}+b_ke^{-\delta_kt_k}\right)\cdot\sum_{k_1,\dots,k_n=-\infty}^{\infty}$$
$$\prod_{i=1}^{n}\frac{\xi\left(\frac{k_i\pi}{\alpha_i}\right)}{a_ie^{\delta_i\frac{k_i\pi}{\alpha_i}}+b_ie^{-\delta_i\frac{k_i\pi}{\alpha_i}}}\cdot\frac{\sin\alpha_i\left(t_i-\frac{k_i\pi}{\alpha_i}\right)}{\alpha_i\left(t_i-\frac{k_i\pi}{\alpha_i}\right)}\cdot\left(\frac{\sin\beta_i\left(t_i-\frac{k_i\pi}{\alpha_i}\right)}{\beta_i\left(t_i-\frac{k_i\pi}{\alpha_i}\right)}\right)^{q_i},$$

for every $\alpha_i>\sigma_i,\ 0<\beta_i<\frac{\alpha_i-\sigma_i}{q_i}$,

$0<\delta_i<\alpha_i-\sigma_i-q_i\beta_i$, where q_i are fixed nonnegative integers, a_i,b_i are positive real numbers (when $q_i=0$, then $\beta_i=0$), $i=1,2,\dots,n$. Definition of σ_i is given in [4].

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