On Estimation of the Remainder Term of the Generalized Sampling Series

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ABSTRACT. In the present paper estimates of the remainder term of the generalized sampling series are considered. Their comparisons for the small and large numerical values of the included $\beta$ parameter are given.

The generalized sampling formula is given in [1]. The proof of this formula is analogous to the theorem 1 from [2] and is based on the following estimate: if $f(z)$ is an entire function, which satisfies the condition

$$|f(z)| \leq L_f(1 + |z|^\mu)e^{\rho|y|}, \quad z = x + iy,$$

(1)

for some non-negative integer $m$ and the positive real numbers $L_f$ and $\sigma$ [2], then for the fixed arbitrary $z$ and for all sufficiently large positive integer $n$ we shall have

$$\left[ \frac{1}{n!} \lim_{z \to 0} \frac{d^n}{dz^n} \left( \frac{f(z)}{(ze^{i\kappa} + be^{-i\kappa}) \sin \left( \frac{\beta(\zeta - z)}{\beta(z - \zeta)} \right)} \right) \right]^{N+1} - \sum_{k=-n}^{N} \frac{(-1)^k}{n!(z^n - \zeta^n)}.$$

$$+ \sum_{k=0}^{N} \frac{\Psi_{1N}(z; k, q, a, \beta, a, b, \beta)}{(N - \tau)!} \frac{\alpha(z - k\pi \alpha)}{(\tau - \mu)!}.$$

$$+ A_{\mu \nu N} \sum_{j=0}^{\mu} B_{\mu j} \left( z - \frac{k\pi \alpha}{\alpha} \right)^j \frac{f^{(j)}(\frac{k\pi \alpha}{\alpha})}{\alpha} \left( \frac{1}{(z - k\pi \alpha)^\nu} \right) \leq L_f \cdot C_{\mu \nu N}(z; a, b, \beta).$$

$$\cdot e^{-\frac{1}{2}(N+1)\pi \sigma^2} \left[ \frac{\alpha}{\pi n^{1/2}} \right]^{p+q+1} + \frac{\alpha}{\pi (n+1)^{1/2}} \left[ \frac{\alpha}{\pi n^{1/2}} \right]^{p+q+1-m}.$$

The function $C_{\mu \nu N}(z; a, b, \beta) = \frac{p+q+1}{\nu \nu} \left( \frac{\pi}{1-e^{-\pi}} \right)^{N+1}$, where $D_0(a, b) = D_0(b, a) = \min(a, b, |a - b|)$, is finite on the arbitrarily bounded domain of the variable $z$. (when $\delta > 0$, then we have “exponential” convergence.)

Note that in the right hand side of inequality (3) from [1] (respectively in inequality (5) from [1]) the multiplier \( \theta^\alpha = \frac{1}{(N+1)\alpha - \sigma - \delta - \beta} \) is omitted. The estimate (2) can be obtained if we apply Cauchy’s residuals theorem for the integral
\[
\int \frac{f(\xi)}{\| \xi - z \|^q} \, \text{d} \xi,
\]
and estimate this integral, where \( C_n \) is circle \( |\xi| = \frac{n}{a} + \frac{1}{2} \). To obtain the estimate (2) the following inequality is used:
\[
\left| \frac{\sin \theta}{\theta} \right| \leq e^{\theta/|\theta|}, \quad z = x + iy.
\]
(3)

The inequality
\[
\left| \frac{\sin \theta}{\theta} \right| \leq e^{\theta/|\theta|}, \quad z = x + iy
\]
is correct also. The inequality (4) with \( |\theta| > 1 \) is rougher than (3), while with \( |\theta| < 1 \) inequality (4) is softer than (3). If we apply the inequality (4), then instead of (2) we shall obtain the following estimate:
\[
\frac{1}{p!} \left( \frac{f(\xi)}{(a \theta + b \theta - \delta \theta)} \sin^{N+1}(\alpha \theta) \right) - \sum_{k=-n}^{n} \left( \frac{(-1)^k}{a \theta^k} \right) \left( \left( \frac{a}{p!} \right)^{N+1} \right) \leq
\]
\[
N \sum_{n=0}^{N} \varphi_n(x; k, a, \alpha, \beta, \mu, \delta) \left( \sum_{\mu=0}^{N} \frac{A_{\alpha \mu N}}{(\tau + \mu) N^\mu} \cdot B_{p \mu} \left( \frac{z - k \alpha N^\mu}{\alpha \theta} \right) \right) \cdot \left( \frac{1}{Z - k \alpha N^\mu} \right)^p \leq
\]
\[
N \sum_{n=0}^{N} \varphi_n(x; k, a, \alpha, \beta, \mu, \delta) \left( \sum_{\mu=0}^{N} \frac{A_{\alpha \mu N}}{(\tau + \mu) N^\mu} \cdot B_{p \mu} \left( \frac{z - k \alpha N^\mu}{\alpha \theta} \right) \right) \cdot \left( \frac{1}{Z - k \alpha N^\mu} \right)^p \leq
\]
\[
L_f \cdot C_{pqN}^0 (z; a, \alpha, \beta) \cdot e^{-\delta (\frac{1}{N+1})} \left( \frac{a}{\pi (\frac{1}{N+1})} \right)^p \left( \frac{a}{\pi (\frac{1}{N+1})} \right)^{p-m} \]
\[
(5)
\]
where \( C_{pqN}^0 (z; a, \alpha, \beta) = \frac{2^{p+1} e^{\delta \theta |\theta|}}{D_{p,q} (a, b)} \left( \frac{2}{1-x} \right)^{N+1} \). So we see that the estimate (5) is suitable for small values of parameter \( \beta \), even at \( \beta = 0 \), however, in this case the estimate (2) makes no sense. When \( \beta \to 0 \) then the right hand side of the inequality (2) converges to infinity, and the right hand side of the inequality (5) converges to the value
\[
L_f \cdot C_{pqN}^0 (z; a, b, 0) \cdot e^{-\delta (\frac{1}{N+1})} \left( \frac{a}{\pi (\frac{1}{N+1})} \right)^p \left( \frac{a}{\pi (\frac{1}{N+1})} \right)^{p-m} \]
\[
(5)
\]
where \( C_{pqN}^0 (z; a, b, 0) = \frac{2^{p+1} e^{\delta \theta |\theta|}}{D_{p,q} (a, b)} \left( \frac{2}{1-x} \right)^{N+1} \).

It should be noted that under specific conditions formula (1) from [1] is true in a certain sense for \( L^p (\Omega) \) random processes also (see [3]). By analogy with the formula (4) from [1], for n-dimensional random field we have
\[
\xi_{(n+1)} = \prod_{k=1}^{n} \left( a_k e^{i\theta_k} + b_k e^{-i\theta_k} \right) \cdot \sum_{k=1}^{\infty} a_k e^{i\theta_k} \cdot \frac{\sin \beta_{(n+1)}}{\beta_{(n+1)}} \left( \frac{\sin \beta_{(n+1)}}{\beta_{(n+1)}} \right)^q \]
\[
(5)
\]
for every \( \alpha_i > \sigma_i > 0 < \beta_i < \frac{\alpha_i - \sigma_i}{q_i} \).
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REFERENCES


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