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*Mathematical Physics*

# On the Investigation of One-Dimensional Models for Thermoelastic Beams

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**ABSTRACT. In the present paper an initial-boundary value problem for thermoelastic beam is considered. Three-dimensional dynamical problem for beam with clamped upper butt end and surface forces given along the remaining part of the boundary is reduced to a hierarchy of one-dimensional problems. The obtained problems are investigated in suitable function spaces, the convergence of the sequence of vector-functions of three space variables, restored from the solutions of one-dimensional problems, to the solution of the original three-dimensional problem is proved and the rate of approximation is estimated.** *© 2009 Bull. Georg. Natl. Acad. Sci.*

***Key words:*** *dynamical problems for thermoelastic beams, double Fourier-Legendre series.*

Mathematical models for elastic bodies defined on two-dimensional and one-dimensional space domains are widely used for designing various engineering structures. In the theory of elasticity, I. Vekua suggested a dimensional reduction method for plates with variable thickness in [1]. According to this method components of the displacement vector-function are expanded into orthogonal Fourier-Legendre series with respect to the variable of plate thickness and considering only the first $N+1$ terms a hierarchy of differential two-dimensional models was constructed. The relationship between three-dimensional and the constructed hierarchical model for plate was studied in [3], and two-dimensional boundary value problems for general shells were first investigated in [4]. Further, mathematical models obtained by I. Vekua's reduction method, its generalizations and various issues of hierarchical modeling were studied in the papers [5-12].

The present paper is devoted to the construction and investigation of one-dimensional hierarchical models of beam, taking into account thermal properties of the elastic material. We consider the variational formulation of the three-dimensional initial-boundary value problem for linearly thermoelastic beam and construct its one-dimensional hierarchical model in Sobolev spaces, when the temperature vanishes along the entire boundary of the body, the upper butt end is clamped and surface forces are given along the remaining part of the boundary of the beam. We investigate the existence and uniqueness of solutions of the reduced one-dimensional problems in suitable weighted Sobolev spaces. Moreover, we prove the convergence of the sequence of vector-functions of three space variables to the solution of the original three-dimensional problem and under additional regularity conditions we estimate the rate of convergence.

Let $\Omega \subset \mathbf{R}^p, p \geq 1,$ be a bounded domain with Lipschitz boundary. $L^2(\Omega)$ denotes the space of square-integrable functions in $\Omega$ in the Lebesgue sense. $W^{k,2}(\Omega) = H^k(\Omega),\ \ k \geq 1,$ is the Sobolev space of order $k$ based on $L^2(\Omega)$, $\mathbf{H}^k(\Omega) = \left[H^k(\Omega)\right]^3$, $\mathbf{L}^2(\Omega) = \left[L^2(\Omega)\right]^3$. We denote by $H_0^k(\Omega)$ the closure of the set $D(\Omega)$ of infinitely differentiable functions with compact support in $\Omega$ in the space $H^k(\Omega)$, $k \geq 1$. For any Banach space $X$,

$C^0([0,T];X)$ denotes the space of continuous functions on $[0,T]$ with values in $X$, $L^2(0,T;X)$ is the space of such functions $g:(0,T)\to X$ that $\|g(t)\|_X \in L^2(0,T)$. We denote by $g' = dg/dt$ the generalized derivative of $g \in L^2(0,T;X)$.

Let us consider a thermoelastic beam with variable rectangular cross-section with possibly vanishing thickness or width on the lower butt end, i.e. the initial configuration of the beam is a Lipschitz domain $\Omega$ of the following form

$$\Omega = \left\{(x_1,x_2,x_3)\in \mathbf{R}^3;\ h_1^-(x_3) < x_1 < h_1^+(x_3),\ h_2^-(x_3) < x_2 < h_2^+(x_3),\ x_3 \in I = (h_3^-,h_3^+)\right\},$$

where $h_1^{\pm}, h_2^{\pm} \in C^0(\bar{I})\cap Lip_{loc}(I)$ are locally Lipschitz continuous in $I$, $h_1^+(x_3) > h_1^-(x_3)$, $h_2^+(x_3) > h_2^-(x_3)$, for $x_3 \in (h_3^-, h_3^+]$, $h_1^{\pm}(x_3)$ and $h_2^{\pm}(x_3)$ are equal or different for $x_3 = h_3^-$. The upper butt end of the beam $\Omega$, defined by the equation $x_3 = h_3^+$ we denote by $\Gamma_0 = \{x \in \mathbf{R}^3; h_\alpha^-(x_3) < x_\alpha < h_\alpha^+(x_3), \alpha = 1,2, x_3 = h_3^+\}$ and the remaining part of the boundary we denote by $\Gamma_1$.

We study an initial-boundary value problem for a thermoelastic beam which consists of homogeneous anisotropic thermoelastic material with elastic constants $\mu_{ijpq} = \mu_{pqij} = \mu_{jipq}$, $i, j, p, q = 1,2,3,$ density $\rho > 0$, heat conductivity coefficients $\lambda_{pq} = \lambda_{qp}$, $p,q = 1,2,3,$ thermal capacity $\chi > 0$ and thermoelastic constants $\eta_{pq} = \eta_{qp}$, $p,q = 1,2,3,$ which define mechanical and thermal properties of the medium. We denote the applied body force density by $\mathbf{f} = (f_i):\Omega\times(0,T) \to \mathbf{R}^3$ and the density of heat sources by $f^\theta : \Omega\times(0,T)\to \mathbf{R}$. We assume that the temperature $\theta$ vanishes along the entire boundary $\Gamma = \partial\Omega$ of the domain $\Omega$, the beam is clamped along the upper butt end $\Gamma_0$ and on the remaining part $\Gamma_1 = \Gamma \setminus \overline{\Gamma_0}$ of the boundary surface force with density $\mathbf{g} = (g_i):\Gamma_1\times(0,T)\to \mathbf{R}^3$ is given.

The dynamical linear three-dimensional model of stress-strain state of homogeneous anisotropic thermoelastic body in differential form is given by

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(\sum_{p,q=1}^{3}\mu_{ijpq}e_{pq}(\mathbf{u}) - \eta_{ij}\theta\right) + f_i \qquad \text{in } \Omega,\ i = 1,2,3, \tag{1}$$

$$\chi\frac{\partial\theta}{\partial t} = \sum_{p,q=1}^{3}\frac{\partial}{\partial x_p}\left(\lambda_{pq}\frac{\partial\theta}{\partial x_q}\right) - \Theta_0\frac{\partial}{\partial t}\sum_{p,q=1}^{3}\eta_{pq}e_{pq}(\mathbf{u}) + f^\theta \qquad \text{in } \Omega, \tag{2}$$

$$\sum_{j=1}^{3}\left(\sum_{p,q=1}^{3}\mu_{ijpq}e_{pq}(\mathbf{u}) - \eta_{ij}\theta\right)\nu_j = g_i \quad \text{on } \Gamma_1,\quad \mathbf{u}(0) = \mathbf{0} \text{ on } \Gamma_0,\quad \theta = 0 \text{ on } \Gamma, \tag{3}$$

$$u_i(x,0) = u_{0i}(x),\quad \frac{\partial u_i}{\partial t}(x,0) = u_{1i}(x),\quad \theta(x,0) = \theta_0(x) \quad \text{in } \Omega,\ i = 1,2,3, \tag{4}$$

where $\Theta_0 > 0$ is the temperature of the medium in the natural state, $e_{ij}(\mathbf{v}) = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$, $i,j = 1,2,3$, $\nu = (\nu_i)$ is the outward unit normal to $\Gamma_1$, $\mathbf{u} = (u_i):\Omega\times(0,T)\to\mathbf{R}^3$ is the displacement vector-function of thermoelastic body, $\theta:\Omega\times(0,T)\to\mathbf{R}$ is the temperature distribution. Multiplying the corresponding equations (1) by enough smooth functions $v_i$, which vanish along $\Gamma_0$, equation (2) by enough smooth function $\varphi$ vanishing along $\Gamma$, integrating the obtained equations over the domain $\Omega$, applying Green's formula and taking into account boundary conditions (3), we obtain

$$\begin{gathered}\sum_{i=1}^{3}\int_\Omega \rho\frac{\partial^2 u_i}{\partial t^2}v_i dx + \int_\Omega\sum_{i,j,p,q=1}^{3}\mu_{ijpq}e_{pq}(\mathbf{u})e_{ij}(\mathbf{v})dx + \sum_{i,j=1}^{3}\int_\Omega \eta_{ij}\frac{\partial\theta}{\partial x_j}v_i dx = \sum_{i=1}^{3}\int_\Omega f_i(x)v_i(x)dx + \sum_{i=1}^{3}\int_{\Gamma_1} g_i v_i d\Gamma,\\ \int_\Omega \chi\frac{\partial\theta}{\partial t}\varphi dx + \sum_{p,q=1}^{3}\int_\Omega \lambda_{pq}\frac{\partial\theta}{\partial x_q}\frac{\partial\varphi}{\partial x_p}dx - \Theta_0\sum_{p,q=1}^{3}\int_\Omega \eta_{pq}\frac{\partial u_q}{\partial t}\frac{\partial\varphi}{\partial x_p}dx = \int_\Omega f^\theta(x)\varphi(x)dx,\end{gathered} \tag{5}$$

for all $\mathbf{v}=(v_i)$, which are smooth enough and equal to zero along $\Gamma_0$, and for all sufficiently regular $\varphi$ vanishing along the boundary $\Gamma$. Note that if $\mathbf{u}=(u_i)$ and $\theta$ are enough smooth solutions of the equations (5), then they also satisfy differential equations (1), (2) and boundary conditions (3). So, the problem (1)-(4) is equivalent to the problem (5), (4), which can be used to define the weak solution of the initial-boundary value problem for an anisotropic thermoelastic beam.

In the present paper we employ the following variational formulation of the three-dimensional initial-boundary value problem (1)-(4): Find $\mathbf{u}\in C^0([0,T];\mathbf{V}(\Omega))$, $\mathbf{u}'\in C^0([0,T];\mathbf{L}^2(\Omega))$, $\theta\in L^2(0,T;H_0^1(\Omega))$, which satisfies the following equations in the sense of distributions in $(0,T)$

$$\frac{d}{dt}(\rho\mathbf{u}'(.),\mathbf{v})_{\mathbf{L}^2(\Omega)}+a(\mathbf{u}(.),\mathbf{v})+\sum_{i,j=1}^{3}(\eta_{ij}\frac{\partial\theta}{\partial x_j},v_i)_{L^2(\Omega)}=(\mathbf{f},\mathbf{v})_{\mathbf{L}^2(\Omega)}+(\mathbf{g},\mathbf{v})_{\mathbf{L}^2(\Gamma_1)},\quad\forall\mathbf{v}\in\mathbf{V}(\Omega),\tag{6}$$

$$\frac{d}{dt}(\chi\theta(.),\varphi)_{L^2(\Omega)}+a^\theta(\theta(.),\varphi)-\Theta_0\sum_{p,q=1}^{3}(\eta_{pq}u_q',\frac{\partial\varphi}{\partial x_p})_{L^2(\Omega)}=(f^\theta,\varphi)_{L^2(\Omega)},\quad\forall\varphi\in H_0^1(\Omega),\tag{7}$$

together with the initial conditions

$$\mathbf{u}(0)=\mathbf{u}_0,\qquad\mathbf{u}'(0)=\mathbf{u}_1,\qquad\theta(0)=\theta_0,\tag{8}$$

where $\mathbf{u}_0$, $\mathbf{u}_1$ are the initial displacement and velocity vector-functions, $\theta_0$ is the initial distribution of temperature, $\mathbf{u}_0\in\mathbf{V}(\Omega)=\{\mathbf{v}\in\mathbf{H}^1(\Omega);\mathbf{tr}(\mathbf{v})=\mathbf{0}\ \text{ on }\Gamma_0\}$, $\mathbf{tr}$ is the trace operator from $\mathbf{H}^1(\Omega)$ to $\mathbf{H}^{1/2}(\Gamma)$, $(.,.)_{\mathbf{L}^2(\Omega)}$, $(.,.)_{L^2(\Omega)}$ and $(.,.)_{\mathbf{L}^2(\Gamma_1)}$ are scalar products in the spaces $\mathbf{L}^2(\Omega)$, $L^2(\Omega)$ and $\mathbf{L}^2(\Gamma_1)$, respectively,

$$a(\tilde{\mathbf{v}},\mathbf{v})=\int_\Omega\sum_{i,j,p,q=1}^{3}\mu_{ijpq}e_{pq}(\tilde{\mathbf{v}})e_{ij}(\mathbf{v})dx,\qquad\forall\mathbf{v},\tilde{\mathbf{v}}\in\mathbf{V}(\Omega),$$

$$a^\theta(\tilde\varphi,\varphi)=\int_\Omega\sum_{p,q=1}^{3}\lambda_{pq}\frac{\partial\tilde\varphi}{\partial x_q}\frac{\partial\varphi}{\partial x_p}dx,\qquad\forall\varphi,\tilde\varphi\in H_0^1(\Omega).$$

The formulated three-dimensional initial-boundary value problem (6)-(8) possesses a unique solution if $\mathbf{u}_0\in\mathbf{V}(\Omega)$, $\mathbf{u}_1\in\mathbf{L}^2(\Omega)$, $\theta_0\in L^2(\Omega)$, $\mathbf{f}=(f_i)_{i=1}^3\in L^2(0,T;\mathbf{L}^2(\Omega))$, $\mathbf{g},\mathbf{g}'\in L^2(0,T;\ \mathbf{L}^{4/3}(\Gamma_1))$, $f^\theta\in L^2(0,T;L^{6/5}(\Omega))$ and the constants defining mechanical and thermal properties of the medium satisfy the following conditions

$$\begin{aligned}&\sum_{i,j,p,q=1}^{3}\mu_{ijpq}\varepsilon_{pq}\varepsilon_{ij}\ge c_\mu\sum_{i,j=1}^{3}(\varepsilon_{ij})^2,\qquad\forall\varepsilon_{ij}\in\mathbf{R},\ \varepsilon_{ij}=\varepsilon_{ji},i,j=1,2,3,\\&\sum_{p,q=1}^{3}\lambda_{pq}\varepsilon_p\varepsilon_q\ge c_\lambda\sum_{p=1}^{3}(\varepsilon_p)^2,\qquad\forall\varepsilon_p\in\mathbf{R},\ p=1,2,3.\end{aligned}\tag{9}$$

In order to construct the hierarchy of two-dimensional models we approximate the original spaces corresponding to the three-dimensional problem by subspaces consisting of vector-functions whose components are polynomials with respect to the variables $x_1$ and $x_2$. In the case of displacement vector-function we consider the subspaces $\mathbf{V}_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$ and $\mathbf{H}_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$ of $\mathbf{V}(\Omega)$ and $\mathbf{L}^2(\Omega)$, respectively, $\mathbf{N}^\alpha=(N_1^\alpha,N_2^\alpha,N_3^\alpha)$, $\alpha=1,2$, which consist of the following vector-functions

$$\mathbf{v}_{\mathbf{N}^1\mathbf{N}^2}=(v_{\mathbf{N}^1\mathbf{N}^2i}),\quad v_{\mathbf{N}^1\mathbf{N}^2i}=\sum_{r_i^1=0}^{N_i^1}\sum_{r_i^2=0}^{N_i^2}\frac{1}{h_1h_2}(r_i^1+\frac{1}{2})(r_i^2+\frac{1}{2})\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2i}P_{r_i^1}(z_1)P_{r_i^2}(z_2),$$

$$\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2i}\in L^2(I),\ \ 0\le r_i^\alpha\le N_i^\alpha,\ \ i=1,2,3,\ z_\alpha=\frac{x_\alpha-\bar h_\alpha}{h_\alpha},\ h_\alpha=\frac{h_\alpha^+-h_\alpha^-}{2},\ \bar h_\alpha=\frac{h_\alpha^++h_\alpha^-}{2},\ \alpha=1,2.$$

For the temperature field we consider the subspaces $V^\theta_{N_\theta^1N_\theta^2}(\Omega)$ and $H^\theta_{N_\theta^1N_\theta^2}(\Omega)$ of $H_0^1(\Omega)$ and $L^2(\Omega)$, respectively, which consist of polynomials with respect to the variables of beam thickness and width satisfying homogeneous Dirichlet boundary conditions on the entire boundary

$$\varphi_{N_\theta^1 N_\theta^2} = \sum_{r_1=0}^{N_\theta^1}\sum_{r_2=0}^{N_\theta^2}\frac{1}{h_1h_2}(r_1+\frac{1}{2})(r_2+\frac{1}{2})\overset{r_1r_2}{\varphi}_{N_\theta^1N_\theta^2}P_{r_1}(z_1)P_{r_2}(z_2)-$$
$$-\sum_{r_1=0}^{N_\theta^1}\sum_{r_2=0}^{N_\theta^2}\frac{1}{2h_1h_2}(r_1+\frac{1}{2})(r_2+\frac{1}{2})\overset{r_1r_2}{\varphi}_{N_\theta^1N_\theta^2}\sum_{\alpha,\beta=1}^{2}(1+(-1)^{r_{\overline{\alpha}}+N_\theta^{\overline{\alpha}}+\beta})P_{r_\alpha}(z_\alpha)P_{N_\theta^{\overline{\alpha}}+\beta}(z_{\overline{\alpha}})+$$
$$+\sum_{r_1=0}^{N_\theta^1}\sum_{r_2=0}^{N_\theta^2}\frac{1}{4h_1h_2}(r_1+\frac{1}{2})(r_2+\frac{1}{2})\overset{r_1r_2}{\varphi}_{N_\theta^1N_\theta^2}\sum_{\alpha,\beta=1}^{2}(1+(-1)^{r_1+N_\theta^1+\alpha})(1+(-1)^{r_2+N_\theta^2+\beta})P_{N_\theta^1+\alpha}(z_1)P_{N_\theta^2+\beta}(z_2),$$
$$\overset{r_1r_2}{\varphi}_{N_\theta^1N_\theta^2}\in L^2(I),\qquad 0\le r_1\le N_\theta^1,\ 0\le r_2\le N_\theta^2,\ \overline{\alpha}=3-\alpha.$$

Since the functions $h_1^{\pm}$ and $h_2^{\pm}$ defining the geometry of the beam are Lipschitz continuous in the interior of the interval $I$, applying Rademacher's theorem [13], we obtain that $h_1^{\pm}$ and $h_2^{\pm}$ are differentiable almost everywhere in $I$ and $\partial_3 h_1^{\pm}$, $\partial_3 h_2^{\pm}\in L^\infty(I^*)$, for all $I^*=(h_3^{-,*},h_3^{+,*})$, $h_3^-<h_3^{-,*}<h_3^{+,*}<h_3^+$. Therefore, taking into account the positiveness of $h_1^{\pm}$ and $h_2^{\pm}$ in $I$, we have that for any vector-function $\mathbf{v}_{\mathbf{N}^1\mathbf{N}^2}=(v_{\mathbf{N}^1\mathbf{N}^2 i})_{i=1}^3\in\mathbf{V}_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$ the corresponding functions $\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2 i}$ belong to $H^1(I^*)$ for all $I^*$, $\overline{I^*}\subset I$, i.e. $\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2 i}\in H^1_{loc}(I)$, $0\le r_i^1\le N_i^1$, $0\le r_i^2\le N_i^2$, $i=1,2,3$. Similarly, for any function $\varphi_{N_\theta^1N_\theta^2}\in V^\theta_{N_\theta^1N_\theta^2}(\Omega)$ the functions $\overset{r_1r_2}{\varphi}_{N_\theta^1N_\theta^2}$ of one space variable in the expression of $\varphi_{N_\theta^1N_\theta^2}$ belong to $H^1(I^*)$, $\overline{I^*}\subset I$, i.e. $\overset{r_1r_2}{\varphi}_{N_\theta^1N_\theta^2}\in H^1_{loc}(I)$, $0\le r_1\le N_\theta^1$, $0\le r_2\le N_\theta^2$. Moreover, the norms $\|.\|_{\mathbf{H}^1(\Omega)}$ and $\|.\|_{H^1(\Omega)}$ in the spaces $\mathbf{H}^1(\Omega)$ and $H^1(\Omega)$ define the weighted norms $\|.\|_*$ and $\|.\|_{\theta*}$ of vector-functions $\vec{v}_{\mathbf{N}^1\mathbf{N}^2}\in[H^1_{loc}(I)]^{N^{1,2}_{1,2,3}}$, $N^{1,2}_{1,2,3}=\sum_{i=1}^{3}(N_i^1+1)(N_i^2+1)$, with components $\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2 i}$, $\vec{v}_{\mathbf{N}^1\mathbf{N}^2}=(\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2 i})$, and $\vec{\varphi}_{N_\theta^1N_\theta^2}\in[H^1_{loc}(I)]^{N_\theta^{1,2}}$, $N_\theta^{1,2}=(N_\theta^1+1)(N_\theta^2+1)$, with components $\overset{r_1r_2}{\varphi}_{N_\theta^1N_\theta^2}$, $\vec{\varphi}_{N_\theta^1N_\theta^2}=(\overset{r_1r_2}{\varphi}_{N_\theta^1N_\theta^2})$, such that $\|\vec{v}_{\mathbf{N}^1\mathbf{N}^2}\|_*=\|\mathbf{v}_{\mathbf{N}^1\mathbf{N}^2}\|_{\mathbf{H}^1(\Omega)}$ and $\|\vec{\varphi}_{N_\theta^1N_\theta^2}\|_{\theta*}=\|\varphi_{N_\theta^1N_\theta^2}\|_{H^1(\Omega)}$. Using the properties of the Legendre polynomials [14], we obtain explicit expressions of the norms $\|.\|_*$ and $\|.\|_{\theta*}$,

$$\|\vec{v}_{\mathbf{N}^1\mathbf{N}^2}\|_*^2=\sum_{i=1}^{3}\sum_{r_i^1=0}^{N_i^1}\sum_{r_i^2=0}^{N_i^2}(r_i^1+\frac{1}{2})(r_i^2+\frac{1}{2})\left[\left\|\frac{\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2 i}}{\sqrt{h_1h_2}}\right\|^2_{L^2(I)}+\sum_{\alpha=1}^{2}\left\|\sum_{k_i^\alpha=r_i^\alpha}^{N_i^\alpha}(k_i^\alpha+\frac{1}{2})\frac{1-(-1)^{k_i^\alpha+r_i^\alpha}}{h_1h_2\sqrt{h_\alpha}}\times\right.\right.$$
$$\left.\times\left((2-\alpha)\overset{k_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2 i}+(\alpha-1)\overset{r_i^1k_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2 i}\right)\right\|^2_{L^2(I)}+\left\|\frac{1}{\sqrt{h_1h_2}}\left(\partial_3\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2 i}-\sum_{\alpha=1}^{2}\frac{\partial_3h_\alpha}{h_\alpha}(r_i^\alpha+1)\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2 i}-\right.\right.$$
$$\left.\left.\left.-\sum_{\alpha=1}^{2}\sum_{k_i^\alpha=r_i^\alpha+1}^{N_i^\alpha}(k_i^\alpha+\frac{1}{2})\frac{\partial_3(h_\alpha^+-(-1)^{k_i^\alpha-r_i^\alpha}h_\alpha^-)}{h_\alpha}\left((2-\alpha)\overset{k_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2 i}+(\alpha-1)\overset{r_i^1k_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2 i}\right)\right)\right\|^2_{L^2(I)}\right],$$

$$\|\vec{\varphi}_{N_\theta^1N_\theta^2}\|_{\theta*}^2=\sum_{r_1=0}^{N_\theta^1+2}\sum_{r_2=0}^{N_\theta^2+2}(r_1+\frac{1}{2})(r_2+\frac{1}{2})\left[\left\|\frac{\overset{r_1r_2}{\varphi}_{N_\theta^1N_\theta^2}}{\sqrt{h_1h_2}}\right\|^2_{L^2(I)}+\sum_{\alpha=1}^{2}\left\|\sum_{k_\alpha=r_\alpha}^{N_\theta^\alpha+2}(k_\alpha+\frac{1}{2})\frac{1-(-1)^{k_\alpha+r_\alpha}}{h_1h_2\sqrt{h_\alpha}}\times\right.\right.$$
$$\left.\times\left((2-\alpha)\overset{k_1r_2}{\varphi}_{N_\theta^1N_\theta^2}+(\alpha-1)\overset{r_1k_2}{\varphi}_{N_\theta^1N_\theta^2}\right)\right\|^2_{L^2(I)}+\left\|\frac{1}{\sqrt{h_1h_2}}\left(\partial_3\overset{r_1r_2}{\varphi}_{N_\theta^1N_\theta^2}-\sum_{\alpha=1}^{2}\frac{\partial_3h_\alpha}{h_\alpha}(r_\alpha+1)\overset{r_1r_2}{\varphi}_{N_\theta^1N_\theta^2}-\right.\right.$$

$$-\sum_{\alpha=1}^{2}\sum_{k_\alpha=r_\alpha+1}^{N_\theta^\alpha+2}(k_\alpha+\frac{1}{2})\frac{\partial_3(h_\alpha^+-(-1)^{k_\alpha-r_\alpha}h_\alpha^-)}{h_\alpha}\left((2-\alpha)\overset{k_1r_2}{\varphi}_{N_\theta^1N_\theta^2}+(\alpha-1)\overset{r_1k_2}{\varphi}_{N_\theta^1N_\theta^2}\right)\Bigg\|^2_{L^2(I)}\Bigg],$$

where we assume that the sum with the lower limit greater than the upper one equals zero, and $\overset{r_1r_2}{\varphi}_{N_\theta^1N_\theta^2}=-\sum_{k_\alpha=0}^{N_\theta^\alpha}\frac{k_\alpha+\beta+\frac{1}{2}}{2\left(N_\theta^\alpha+\beta+\frac{1}{2}\right)}\left(1+(-1)^{k_\alpha+N_\theta^\alpha+\beta}\right)\overset{k_1k_2}{\varphi}_{N_\theta^1N_\theta^2}$, $\beta=1,2,$ where $\alpha=1$ and $k_2=r_2$, if $r_1>N_1$, and $\alpha=2$ and $k_1=r_1$, if $r_2>N_2$.

Note that for components $\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2i}$ and $\overset{r_1r_2}{\varphi}_{N_\theta^1N_\theta^2}$ of $\vec{v}_{\mathbf{N}^1\mathbf{N}^2}\in[H^1_{loc}(\omega)]^{N^{1,2}_{1,2,3}}$ and $\vec{\varphi}_{N_\theta^1N_\theta^2}\in[H^1_{loc}(\omega)]^{N_\theta^{1,2}}$, which possess the properties $\left\|\vec{v}_{\mathbf{N}^1\mathbf{N}^2}\right\|_*<\infty$ and $\left\|\vec{\varphi}_{N_\theta^1N_\theta^2}\right\|_{\theta*}<\infty$ we can define the trace for $x_3=h_3^+$. Indeed, the corresponding vector-function of three space variables $\mathbf{v}_{\mathbf{N}^1\mathbf{N}^2}=(v_{\mathbf{N}^1\mathbf{N}^2i})_{i=1}^3$ and function $\varphi_{N_\theta^1N_\theta^2}$ belong to the space $\mathbf{V}_{\mathbf{N}^1\mathbf{N}^2}(\Omega)\subset\mathbf{H}^1(\Omega)$ and $V^\theta_{N_\theta^1N_\theta^2}(\Omega)\subset H^1(\Omega)$, respectively. Hence, applying the trace operator $tr:H^1(\Omega)\to H^{1/2}(\Gamma)$, for any $\overset{r_1r_2}{\varphi}_{N_\theta^1N_\theta^2}$ we define the trace for $x_3=h_3^+$,

$$tr_{\theta*}(\overset{r_1r_2}{\varphi}_{N_\theta^1N_\theta^2})=\int_{h_1^-}^{h_1^+}\int_{h_2^-}^{h_2^+}tr(\varphi_{N_\theta^1N_\theta^2})|_{\Gamma_0}P_{r_1}(z_1)P_{r_2}(z_2)dx_1dx_2,\quad 0\le r_\alpha\le N_\theta^\alpha,\quad \alpha=1,2,$$

and similarly we define the trace $tr_*(\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2i})$ for $x_3=h_3^+$, $0\le r_i^1\le N_i^1$, $0\le r_i^2\le N_i^2$, $i=1,2,3$.

Since the vector-functions $\mathbf{v}_{\mathbf{N}^1\mathbf{N}^2}=(v_{\mathbf{N}^1\mathbf{N}^2i})_{i=1}^3$ from the subspaces $\mathbf{V}_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$ and $\mathbf{H}_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$, and the functions $\varphi_{N_\theta^1N_\theta^2}$ from $V^\theta_{N_\theta^1N_\theta^2}(\Omega)$ and $H^\theta_{N_\theta^1N_\theta^2}(\Omega)$ are defined by functions $\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2i}$ and $\overset{r_1r_2}{\varphi}_{N_\theta^1N_\theta^2}$ of one space variable, on the introduced subspaces of the original spaces we obtain the hierarchy of one-dimensional problems corresponding to the original three-dimensional problem (6)-(8): Find $\vec{w}_{\mathbf{N}^1\mathbf{N}^2}\in C^0([0,T];\vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I))$, $\vec{w}'_{\mathbf{N}^1\mathbf{N}^2}\in C^0([0,T];\vec{H}_{\mathbf{N}^1\mathbf{N}^2}(I))$, $\vec{\zeta}_{N_\theta^1N_\theta^2}\in L^2(0,T;\vec{V}^\theta_{N_\theta^1N_\theta^2}(I))$, which satisfy the following equations in the sense of distributions in $(0,T)$

$$\frac{d}{dt}R_{\mathbf{N}^1\mathbf{N}^2}(\vec{w}'_{\mathbf{N}^1\mathbf{N}^2},\vec{v}_{\mathbf{N}^1\mathbf{N}^2})+a_{\mathbf{N}^1\mathbf{N}^2}(\vec{w}_{\mathbf{N}^1\mathbf{N}^2},\vec{v}_{\mathbf{N}^1\mathbf{N}^2})+b_{\mathbf{N}^1\mathbf{N}^2N_\theta^1N_\theta^2}(\vec{\zeta}_{N_\theta^1N_\theta^2},\vec{v}_{\mathbf{N}^1\mathbf{N}^2})=L_{\mathbf{N}^1\mathbf{N}^2}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2}),\tag{10}$$

$$\frac{d}{dt}R^\theta_{N_\theta^1N_\theta^2}(\vec{\zeta}_{N_\theta^1N_\theta^2},\vec{\varphi}_{N_\theta^1N_\theta^2})+a^\theta_{N_\theta^1N_\theta^2}(\vec{\zeta}_{N_\theta^1N_\theta^2},\vec{\varphi}_{N_\theta^1N_\theta^2})-\Theta_0b^\theta_{\mathbf{N}^1\mathbf{N}^2N_\theta^1N_\theta^2}(\vec{w}'_{\mathbf{N}^1\mathbf{N}^2},\vec{\varphi}_{N_\theta^1N_\theta^2})=L^\theta_{N_\theta^1N_\theta^2}(\vec{\varphi}_{N_\theta^1N_\theta^2}),\tag{11}$$

for all $\vec{v}_{\mathbf{N}^1\mathbf{N}^2}\in\vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I)$ and $\vec{\varphi}_{N_\theta^1N_\theta^2}\in\vec{V}^\theta_{N_\theta^1N_\theta^2}(I)$, together with the initial conditions

$$\vec{w}_{\mathbf{N}^1\mathbf{N}^2}(0)=\vec{w}_{\mathbf{N}^1\mathbf{N}^20},\qquad \vec{w}'_{\mathbf{N}^1\mathbf{N}^2}(0)=\vec{w}_{\mathbf{N}^1\mathbf{N}^21},\qquad \vec{\zeta}_{N_\theta^1N_\theta^2}(0)=\vec{\zeta}_{N_\theta^1N_\theta^20},\tag{12}$$

where $\vec{w}_{\mathbf{N}^1\mathbf{N}^20}\in\vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I)=\{\vec{v}_{\mathbf{N}^1\mathbf{N}^2}=(\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2i})\in[H^1_{loc}(I)]^{N^{1,2}_{1,2,3}};\left\|\vec{v}_{\mathbf{N}^1\mathbf{N}^2}\right\|_*<\infty,tr_*(\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2i})=0,\quad 0\le r_i^\alpha\le N_i^\alpha,\quad \alpha=1,2,$ $i=1,2,3\}$, $\vec{w}_{\mathbf{N}^1\mathbf{N}^21}\in\vec{H}_{\mathbf{N}^1\mathbf{N}^2}(I)=\{\vec{v}_{\mathbf{N}^1\mathbf{N}^2}\in[L^2(I)]^{N^{1,2}_{1,2,3}};(h_1h_2)^{-1/2}\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2i}\in L^2(I),\ 0\le r_i^\alpha\le N_i^\alpha,\quad \alpha=1,2,\quad i=1,2,3\}$, $\vec{V}^\theta_{N_\theta^1N_\theta^2}(I)=\{\vec{\varphi}_{N_\theta^1N_\theta^2}=(\overset{r_1r_2}{\varphi}_{N_\theta^1N_\theta^2})\in[H^1_{loc}(I)]^{N_\theta^{1,2}};\left\|\vec{\varphi}_{N_\theta^1N_\theta^2}\right\|_{\theta*}<\infty,tr_{\theta*}(\overset{r_1r_2}{\varphi}_{N_\theta^1N_\theta^2})=0,\quad 0\le r_\alpha\le N_\theta^\alpha,\quad \alpha=1,2\}$,

$\vec{\zeta}_{N_\theta^1N_\theta^20}\in\vec{H}^\theta_{N_\theta^1N_\theta^2}(I)=\{\vec{\varphi}_{N_\theta^1N_\theta^2}\in[L^2(I)]^{N_\theta^{1,2}};(h_1h_2)^{-1/2}\overset{r_1r_2}{\varphi}_{N_\theta^1N_\theta^2}\in L^2(I),\quad 0\le r_\alpha\le N_\theta^\alpha,\alpha=1,2\}$, the bilinear forms $R_{\mathbf{N}^1\mathbf{N}^2}$, $R^\theta_{N_\theta^1N_\theta^2}$, $a_{\mathbf{N}^1\mathbf{N}^2}$, $a^\theta_{N_\theta^1N_\theta^2}$, $b_{\mathbf{N}^1\mathbf{N}^2N_\theta^1N_\theta^2}$, $b^\theta_{\mathbf{N}^1\mathbf{N}^2N_\theta^1N_\theta^2}$ are defined by the corresponding forms in the left-hand sides of the equations (6), (7) and taking into account the properties of the Legendre polynomials, we have

$$R_{\mathbf{N}^1\mathbf{N}^2}(\vec{y}_{\mathbf{N}^1\mathbf{N}^2},\vec{v}_{\mathbf{N}^1\mathbf{N}^2})=\sum_{i=1}^{3}\sum_{r_i^1=0}^{N_i^1}\sum_{r_i^2=0}^{N_i^2}\left(r_i^1+\frac{1}{2}\right)\left(r_i^2+\frac{1}{2}\right)\int_I\frac{\rho}{h_1h_2}\overset{r_i^1r_i^2}{y}_{\mathbf{N}^1\mathbf{N}^2 i}\overset{r_i^1r_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2 i}dx_3,$$

$$a_{\mathbf{N}^1\mathbf{N}^2}(\vec{y}_{\mathbf{N}^1\mathbf{N}^2},\vec{v}_{\mathbf{N}^1\mathbf{N}^2})=\sum_{i,j,p,q=1}^{3}\mu_{ijpq}\sum_{r_1=0}^{N^1_{\max}}\sum_{r_2=0}^{N^2_{\max}}\left(r_1+\frac{1}{2}\right)\left(r_2+\frac{1}{2}\right)\int_I\frac{1}{h_1h_2}\overset{r_1r_2}{e}_{pq}(\vec{y}_{\mathbf{N}^1\mathbf{N}^2})\overset{r_1r_2}{e}_{ij}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2})dx_3,$$

$$b_{\mathbf{N}^1\mathbf{N}^2N_\theta^1N_\theta^2}(\vec{\varphi}_{N_\theta^1N_\theta^2},\vec{v}_{\mathbf{N}^1\mathbf{N}^2})=b^\theta_{\mathbf{N}^1\mathbf{N}^2N_\theta^1N_\theta^2}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2},\vec{\varphi}_{N_\theta^1N_\theta^2})=\sum_{i=1}^{3}\sum_{k_1=0}^{N_\theta^1+2}\sum_{k_2=0}^{N_\theta^2+2}\prod_{\alpha=1}^{2}(k_\alpha+\frac{1}{2})\int_{h_3^-}^{h_3^+}\frac{1}{h_1h_2}\Bigg(\eta_{i3}\Bigg[\partial_3\overset{k_1k_2}{\varphi}_{N_\theta^1N_\theta^2}-$$

$$-\sum_{\alpha=1}^{2}\frac{\partial_3h_\alpha(k_\alpha+1)}{h_\alpha}\overset{k_1k_2}{\varphi}_{N_\theta^1N_\theta^2}-\sum_{\alpha=1}^{2}\sum_{s_\alpha=k_\alpha+1}^{N_\alpha}\frac{\partial_3(h_\alpha^+-(-1)^{k_\alpha+s_\alpha}h_\alpha^-)}{h_\alpha}(s_\alpha+\frac{1}{2})((2-\alpha)\overset{s_1k_2}{\varphi}_{N_\theta^1N_\theta^2}+(\alpha-1)\overset{k_1s_2}{\varphi}_{N_\theta^1N_\theta^2})\Bigg]\times$$

$$\times\overset{k_1k_2}{v}_{\mathbf{N}^1\mathbf{N}^2 i}+\sum_{\alpha=1}^{2}\eta_{i\alpha}\left[\sum_{s_\alpha=k_\alpha}^{N_\theta^\alpha+2}\frac{1}{h_\alpha}(s_\alpha+\frac{1}{2})\left((2-\alpha)\overset{s_1k_2}{\varphi}_{N_\theta^1N_\theta^2}+(\alpha-1)\overset{k_1s_2}{\varphi}_{N_\theta^1N_\theta^2}\right)(1-(-1)^{s_\alpha+k_\alpha})\right]\overset{k_1k_2}{v}_{\mathbf{N}^1\mathbf{N}^2 i}\Bigg)dx_3,$$

where $N^\alpha_{\max}=\max\limits_{1\le i\le 3}N_i^\alpha$, $\alpha=1,2$, $\overset{k_1k_2}{e}_{ij}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2})=\frac{1}{2}(\partial_i(\overset{k_1k_2}{v}_{\mathbf{N}^1\mathbf{N}^2 j})+\partial_j(\overset{k_1k_2}{v}_{\mathbf{N}^1\mathbf{N}^2 i})+\overset{k_1k_2}{d}_{ij}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2})+\overset{k_1k_2}{d}_{ji}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2}))$,

$$\overset{k_1k_2}{d}_{ij}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2})=-\sum_{\alpha=1}^{2}\frac{\partial_i(h_\alpha)}{h_\alpha}(k_\alpha+1)\overset{k_1k_2}{v}_{\mathbf{N}^1\mathbf{N}^2 j}-\sum_{\alpha=1}^{2}\sum_{s_\alpha=k_\alpha+1}^{N^\alpha_{\max}}\frac{1}{h_\alpha}\left(s_\alpha+\frac{1}{2}\right)\partial_i(h_\alpha^+-(-1)^{k_\alpha+s_\alpha}h_\alpha^-)\times$$

$$((2-\alpha)\overset{s_1k_2}{v}_{\mathbf{N}^1\mathbf{N}^2 j}+(\alpha-1)\overset{k_1s_2}{v}_{\mathbf{N}^1\mathbf{N}^2 j})+\frac{(3-i)i}{2}\sum_{s_i=k_i}^{N^i_{\max}}\frac{1}{h_i}\left(s_i+\frac{1}{2}\right)(1-(-1)^{s_i+k_i})((2-i)\overset{s_1k_2}{v}_{\mathbf{N}^1\mathbf{N}^2 j}+(i-1)\overset{k_1s_2}{v}_{\mathbf{N}^1\mathbf{N}^2 j}),$$

$\overset{k_1k_2}{v}_{\mathbf{N}^1\mathbf{N}^2 i}\equiv 0$, $k_1>N_i^1$ or $k_2>N_i^2$, $i=1,2,3$, and similarly we can obtain expressions for $R^\theta_{N_\theta^1N_\theta^2}(\vec{\xi}_{N_\theta^1N_\theta^2},\vec{\varphi}_{N_\theta^1N_\theta^2})=(\chi\xi_{N_\theta^1N_\theta^2},\varphi_{N_\theta^1N_\theta^2})_{L^2(\Omega)}$ and $a^\theta_{N_\theta^1N_\theta^2}(\vec{\xi}_{N_\theta^1N_\theta^2},\vec{\varphi}_{N_\theta^1N_\theta^2})=a^\theta(\xi_{N_\theta^1N_\theta^2},\varphi_{N_\theta^1N_\theta^2})$. The linear forms $L_{\mathbf{N}}$, $L^\theta_{N_\theta}$ are defined by the right-hand sides of the equations (6), (7) and are given by

$$L_{\mathbf{N}^1\mathbf{N}^2}(\vec{v}_{\mathbf{N}^1\mathbf{N}^2})=\sum_{i=1}^{3}\sum_{k_i^1=0}^{N_i^1}\sum_{k_i^2=0}^{N_i^2}\left(k_i^1+\frac{1}{2}\right)\left(k_i^2+\frac{1}{2}\right)\Bigg[\int_{h_3^-}^{h_3^+}\frac{1}{h_1h_2}\overset{k_i^1k_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2 i}\Bigg(\overset{k_i^1k_i^2}{f}_i+\overset{k_i^2}{g_i\big|}_{\Gamma^{1,+}}\gamma^{1,+}+\overset{k_i^1}{g_i\big|}_{\Gamma^{2,+}}\gamma^{2,+}+$$

$$+\overset{k_i^2}{g_i\big|}_{\Gamma^{1,-}}\gamma^{1,-}(-1)^{k_i^1}+\overset{k_i^1}{g_i\big|}_{\Gamma^{2,-}}\gamma^{2,-}(-1)^{k_i^2}\Bigg)dx_3+\sum_{x_3=h_3^-,h_1h_2>0}\frac{1}{h_1h_2}\overset{k_i^1k_i^2}{g}_i\overset{k_i^1k_i^2}{v}_{\mathbf{N}^1\mathbf{N}^2 i}\Bigg],$$

$$L^\theta_{N_\theta^1N_\theta^2}(\vec{\varphi}_{N_\theta^1N_\theta^2})=\sum_{r_1=0}^{N_\theta^1}\sum_{r_2=0}^{N_\theta^2}\left(r_1+\frac{1}{2}\right)\left(r_2+\frac{1}{2}\right)\int_{h_3^-}^{h_3^+}\Bigg[\frac{1}{h_1h_2}\overset{r_1r_2}{f^\theta}\overset{r_1r_2}{\varphi}_{N_\theta^1N_\theta^2}-\frac{1}{2h_1h_2}\sum_{\beta=1}^{2}\Bigg((1+(-1)^{r_2+N_\theta^2+\beta})\overset{r_1,N_\theta^2+\beta}{f^\theta}+$$

$$+(1+(-1)^{r_1+N_\theta^1+\beta})\overset{N_\theta^1+\beta,r_2}{f^\theta}\Bigg)\overset{r_1r_2}{\varphi}_{N_\theta^1N_\theta^2}+\frac{1}{4h_1h_2}\sum_{\alpha,\beta=1}^{2}(1+(-1)^{r_1+N_\theta^1+\alpha})(1+(-1)^{r_2+N_\theta^2+\beta})\overset{N_\theta^1+\alpha,N_\theta^2+\beta}{f^\theta}\overset{r_1r_2}{\varphi}_{N_\theta^1N_\theta^2}\Bigg]dx_3,$$

where $\overset{k_i^\alpha}{g_i\big|}_{\Gamma^{\overline{\alpha},\pm}}=\int_{h_\alpha^-}^{h_\alpha^+}g_i\,|_{\Gamma^{\overline{\alpha},\pm}}(x_\alpha,x_3)P_{k_i^\alpha}(z_\alpha)dx_\alpha$, $\overline{\alpha}=3-\alpha$, $\alpha=1,2$, $g_i\,|_{\Gamma^{1,\pm}}=g_i(h_1^\pm(x_3),x_2,x_3)$,

$g_i\,|_{\Gamma^{2,\pm}}=g_i(x_1,h_2^\pm(x_3),x_3)$, $\gamma^{\alpha,\pm}=\sqrt{1+((h_\alpha^\pm)')^2}$, $\overset{k_i^1k_i^2}{v}=\int_{h_1^-}^{h_1^+}\int_{h_2^-}^{h_2^+}vP_{k_i^1}(z_1)P_{k_i^2}(z_2)dx_1dx_2$, for $v=f_i$ or $v=g_i(x_1,x_2,h_3^-)$,

$0\le k_i^\alpha\le N_i^\alpha$, $i=1,2,3$, $\alpha=1,2$.

For the obtained one-dimensional initial-boundary value problems (10)-(12) the following existence and uniqueness theorem is proved.

**Theorem 1.** *If the functions* $h_1^{\pm}$ *and* $h_2^{\pm}$ *are such that* $\Omega$ *is a Lipschitz domain, conditions (9) are fulfilled,* $\vec{w}_{\mathbf{N}^1\mathbf{N}^2 0} \in \vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I)$, $\vec{w}_{\mathbf{N}^1\mathbf{N}^2 1} \in \vec{H}_{\mathbf{N}^1\mathbf{N}^2}(I)$, $\vec{\zeta}_{N_\theta^1 N_\theta^2 0} \in \vec{H}^{\theta}_{N_\theta^1 N_\theta^2}(I)$, *and the functions* $\overset{k_i^1 k_i^2}{f_i}$, $\overset{r_1 r_2}{f^{\theta}}$, $\overset{r_i}{f_i}$, $\overset{k_i^{3-\alpha}}{g_i}\Big|_{\Gamma^{\alpha,\pm}}$ *are such that* $\frac{1}{\sqrt{h_1 h_2}}\overset{k_i^1 k_i^2}{f_i} \in L^2(0,T;L^2(I))$, $0 \le k_i^{\alpha} \le N_i^{\alpha}$, $\alpha = 1,2$, $i = 1,2,3$,

$$\frac{d^r}{dt^r}\left(\frac{1}{\sqrt[4]{h_1 h_2}}\sum_{\alpha=1}^{2}\left(\overset{k_i^{3-\alpha}}{g_i}\Big|_{\Gamma^{\alpha,+}}\gamma^{\alpha,+} + \overset{k_i^{3-\alpha}}{g_i}\Big|_{\Gamma^{\alpha,-}}\gamma^{\alpha,-}(-1)^{k_i^{\alpha}}\right)\right) \in L^2(0,T;L^{4/3}(I)), \quad r = 0,1,$$

$$\frac{1}{\sqrt[6]{h_1 h_2}}\left(\overset{r_1 r_2}{f^{\theta}} - \frac{1}{2}\sum_{\beta=1}^{2}\left((1+(-1)^{r_2+N_\theta^2+\beta})\overset{r_1, N_\theta^2+\beta}{f^{\theta}} + (1+(-1)^{r_1+N_\theta^1+\beta})\overset{N_\theta^1+\beta, r_2}{f^{\theta}}\right)+\right.$$

$$\left.+\frac{1}{4}\sum_{\alpha,\beta=1}^{2}(1+(-1)^{r_1+N_\theta^1+\alpha})(1+(-1)^{r_2+N_\theta^2+\beta})\overset{N_\theta^1+\alpha, N_\theta^2+\beta}{f^{\theta}}\right) \in L^2(0,T;L^{6/5}(I)), \ 0 \le r_{\alpha} \le N_\theta^{\alpha}, \ \alpha = 1,2,$$

*then dynamical one-dimensional problem (10)-(12) possesses a unique solution.*

In the next theorem we present the results of investigation of the relationship between the constructed one-dimensional hierarchical model and the original three-dimensional problem, but in order to formulate it let us define the following anisotropic weighted Sobolev space

$$H^{s,s,2}_{h_{1,2}^{\pm}}(\Omega) = \{v \in H^1(\Omega);\ \partial_\alpha^{\tilde{k}} v \in H^2(\Omega),\ \partial_3 h_\alpha^{\pm}\partial_1\partial_2\partial_\alpha^{\tilde{k}} v \in L^2(\Omega),\ 0 \le \tilde{k} \le s-2,$$

$$\partial_3 h_\alpha^{\pm}\partial_\alpha^{k} v \in L^2(\Omega),\ 1 \le k \le s,\ \alpha = 1,2\}, \qquad s \in \mathbf{N}, s \ge 2,$$

which is a Hilbert space equipped with the norm

$$\|v\|^2_{H^{s,s,2}_{h_{1,2}^{\pm}}(\Omega)} = \sum_{\alpha=1}^{2}\left(\sum_{r=0}^{s-2}\left(\|\partial_\alpha^r v\|^2_{H^2(\Omega)} + \|\partial_3 h_\alpha^{+}\partial_1\partial_2\partial_\alpha^r v\|^2_{L^2(\Omega)} + \|\partial_3 h_\alpha^{-}\partial_1\partial_2\partial_\alpha^r v\|^2_{L^2(\Omega)}\right)+\right.$$

$$\left.+\sum_{k=1}^{s}\left(\|\partial_3 h_\alpha^{+}\partial_\alpha^k v\|^2_{L^2(\Omega)} + \|\partial_3 h_\alpha^{-}\partial_\alpha^k v\|^2_{L^2(\Omega)}\right)\right).$$

**Theorem 2.** *If* $\mathbf{u}_0 \in \mathbf{V}(\Omega)$, $\mathbf{u}_1 \in \mathbf{L}^2(\Omega)$, $\theta_0 \in L^2(\Omega)$, $\mathbf{f} = (f_i)_{i=1}^3 \in L^2(0,T;\mathbf{L}^2(\Omega))$, $\mathbf{g},\mathbf{g}' \in L^2(0,T;\ \mathbf{L}^{4/3}(\Gamma_1))$, $f^{\theta} \in L^2(0,T;L^{6/5}(\Omega))$, *conditions (9) are fulfilled and the vector-functions* $\mathbf{w}_{\mathbf{N}^1\mathbf{N}^2 0} \in \mathbf{V}_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$, $\mathbf{w}_{\mathbf{N}^1\mathbf{N}^2 1} \in \mathbf{H}_{\mathbf{N}^1\mathbf{N}^2}(\Omega)$, $\zeta_{N_\theta^1 N_\theta^2 0} \in H^{\theta}_{N_\theta^1 N_\theta^2}(\Omega)$ *corresponding to the initial conditions* $\vec{w}_{\mathbf{N}^1\mathbf{N}^2 0} \in \vec{V}_{\mathbf{N}^1\mathbf{N}^2}(I)$, $\vec{w}_{\mathbf{N}^1\mathbf{N}^2 1} \in \vec{H}_{\mathbf{N}^1\mathbf{N}^2}(I)$, $\vec{\zeta}_{N_\theta^1 N_\theta^2 0} \in \vec{H}^{\theta}_{N_\theta^1 N_\theta^2}(I)$ *of one-dimensional problems, tend to* $\mathbf{u}_0$, $\mathbf{u}_1$ *and* $\theta_0$ *in the spaces* $\mathbf{H}^1(\Omega)$, $\mathbf{L}^2(\Omega)$ *and* $L^2(\Omega)$, *respectively, as* $N_{\min} = \min_{1 \le i \le 3}\{N_i^1, N_i^2, N_\theta^1, N_\theta^2\} \to \infty$, *then the vector-function* $\mathbf{w}_{\mathbf{N}^1\mathbf{N}^2}(t)$ *and function* $\zeta_{N_\theta^1 N_\theta^2}(t)$ *restored from the solutions* $\vec{w}_{\mathbf{N}^1\mathbf{N}^2}$ *and* $\vec{\zeta}_{N_\theta^1 N_\theta^2}$ *of the reduced one-dimensional problem (10)-(12), tend to the solution of the original three-dimensional problem (6)-(8),*

$$\begin{aligned} &\mathbf{w}_{\mathbf{N}^1\mathbf{N}^2}(t) \to \mathbf{u}(t) && \text{in } \mathbf{H}^1(\Omega), \\ &\mathbf{w}'_{\mathbf{N}^1\mathbf{N}^2}(t) \to \mathbf{u}'(t) && \text{in } \mathbf{L}^2(\Omega), \qquad \text{for all } t \in [0,T], \\ &\zeta_{N_\theta^1 N_\theta^2}(t) \to \theta(t) && \text{in } L^2(\Omega), \\ &\zeta_{N_\theta^1 N_\theta^2} \to \theta && \text{in } L^2(0,T;H^1(\Omega)), \quad \text{as } N_{\min} \to \infty. \end{aligned}$$

*In addition, if* $d^p\mathbf{u}/dt^p \in L^2(0,T;\ (H^{s_p,s_p,2}_{h_{1,2}^{\pm}}(\Omega))^3)$, $s_p \in \mathbf{N}$, $p = 0,1,2$, $s_0 \ge s_1 \ge s_2 \ge 2$, $s_1 \ge 3$, $\theta \in L^2(0,T;\ H^{s_0^{\theta},s_0^{\theta},2}_{h_{1,2}^{\pm}}(\Omega))$, $\theta' \in L^2(0,T;\ H^{s_1^{\theta},s_1^{\theta},2}_{h_{1,2}^{\pm}}(\Omega))$, $s_0^{\theta} \ge s_1^{\theta} \ge 2$, $s_0^{\theta} \ge 3$, *then for suitable initial data* $\vec{w}_{\mathbf{N}^1\mathbf{N}^2 0}$, $\vec{w}_{\mathbf{N}^1\mathbf{N}^2 1}$ *and* $\vec{\zeta}_{N_\theta^1 N_\theta^2 0}$ *the following estimate is valid*

$$\left\|\mathbf{u}-\mathbf{w}_{\mathbf{N}^1\mathbf{N}^2}\right\|_{C^0([0,T];\mathbf{H}^1(\Omega))}+\left\|\mathbf{u}'-\mathbf{w}'_{\mathbf{N}^1\mathbf{N}^2}\right\|_{C^0([0,T];\mathbf{L}^2(\Omega))}+$$

$$+\left\|\theta-\zeta_{N_\theta^1N_\theta^2}\right\|_{C^0([0,T];L^2(\Omega))}+\left\|\theta-\zeta_{N_\theta^1N_\theta^2}\right\|_{L^2(0,T;H^1(\Omega))}\leq\frac{1}{(N_{\min})^s}\,o(T,\Omega,\Gamma_0,h_1^\pm,h_2^\pm,\mathbf{N}^1,\mathbf{N}^2,N_\theta^1,N_\theta^2),$$

*where* $s=\min\{s_2,s_1-1,s_1^\theta,s_0^\theta-1\}$, $o(T,\Omega,\Gamma_0,h_1^\pm,h_2^\pm,\mathbf{N}^1,\mathbf{N}^2,N_\theta^1,N_\theta^2)\to 0$, *as* $N_{\min}\to\infty$.

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# თერმოდრეკადი ღეროების ერთგანზომილებიანი მოდელების გამოკვლევის შესახებ


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**ნაშრომში განხილულია საწყის-სასაზღვრო ამოცანა თერმოდრეკადი ღეროსათვის. სამგანზომილებიანი დინამიკური ამოცანა ღეროსათვის, როცა მისი ზედა ტორსული ზედაპირი ჩამაგრებულია, ხოლო საზღვრის დანარჩენ ნაწილზე მოცემულია ძაბვები, დაყვანილია ერთგანზომილებიანი ამოცანების იერარქიაზე. მიღებული ამოცანები გამოკვლეულია შესაბამის ფუნქციონალურ სივრცეებში, დამტკიცებულია ერთგანზომილებიანი ამოცანების ამონახსნებიდან აღდგენილი სამი სივრცითი ცვლადის ვექტორ-ფუნქციათა მიმდევრობის კრებადობა სამგანზომილებიანი ამოცანის ამონახსნისაკენ და მიღებულია კრებადობის რიგის შეფასება.**

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