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Mechanics

On One Mathematical Model of the Linear Oscillator

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ABSTRACT. One mathematical model of the oscillator analysis is given in the paper, which is based on the assumption that, under the pulse force (impulse) action on the oscillator, excitation of the reaction force in the spring (linkage) occurs with an insignificant delay in time. The oscillator motion is represented by differential equations with delayed argument. It is established that the oscillator motion is complex mechanical vibration with damping feature. *© 2009 Bull. Georg. Natl. Acad. Sci.*

Key words: *linear oscillator, damping of vibration, differential equations with delayed argument.*

As is known, a linear oscillator plays an important part in the description of seismic phenomena and thus, each novelty in the issues of oscillator analysis constitutes an important problem [1].

One mathematical model of oscillator analysis is given in the paper, which is based on the assumption that, under the pulse force (impulse) action on the oscillator, excitation of the reaction force in the spring (linkage) occurs with an insignificant delay in time and hence, the motion of the oscillator is represented in two parts: in the period of the impulse action and after the beginning of linkage action. The implication is that, the impulse action durability τ is rather small as compared to the T period of the oscillator free motion: $\tau << T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{m}{k}}$, where m is oscillator mass and k - spring stiffness.

In the period of the impulse action $[0;\ \tau]$ oscillator motion is described by the equation:

$$mv - mv_0 = S\,, \tag{1}$$

where S represents force impulse in the space of time $[0;\ \tau]$:

$$S = \int_0^{\tau} F(t)dt\,;$$

if we consider that $v_0 = 0$, then from (1) we shall have

$$v(\tau) = \frac{S}{m}, \quad x(t) = \frac{S}{m}t\,, \quad t \in [0;\tau]\,. \tag{2}$$

Based on the assumption that linkage action on the oscillator occurs from the time $t \geq \tau$ moment, the elastic reaction force (spring tension) $P(t) = kx(t)$ will be as follows: $P(t) = k\eta(t-\tau)x(t-\tau)$ where $\eta(t-\tau)$ is a Heavyside function and, thus, the differential equation of the oscillator motion will have the following form:

$$x''(t)+\omega^2 x(t-\tau)=0\,;\;\; t\in[\tau;\infty)\,. \tag{3}$$

So, the oscillator motion equation in the space time $[\tau;\infty)$ represents a differential equation with delayed argument [2, 3], the initial function of which is defined from (2).

If we take the designation $\xi=t-\tau$ and carry out Laplace transformation of the equation (3) $\hat{x}(p)=\int\limits_0^\infty x(\xi)\exp(-p\xi)d\xi$ we shall obtain

$$\hat{x}(p)=\frac{S}{m}\frac{e^{P\tau}}{p^2e^{P\tau}+\omega^2}\,. \tag{4}$$

As we have found $\hat{x}(p)$, the function $x(\xi)$ itself will be recovered by Laplace inverse transformation.

$$x(\xi)=\mathrm{Re}[\frac{1}{2\pi i}\lim_{\gamma\to\infty}\int\limits_{\varepsilon-i\gamma}^{\varepsilon+i\gamma}\hat{x}(p)e^{P\xi}dp]. \tag{5}$$

At the same time line $\mathrm{Re}\,p=\varepsilon$ will be chosen so that all the particular points of the function $\hat{x}(p)$ be placed to the left of the mentioned line (based on the displacement limit ($x(\xi)$ condition, under ε we can imply any small positive number).

To calculate the integral included in formula (5) let us close the section $[\varepsilon-i\gamma;\varepsilon+i\gamma]$ located in $\mathrm{Re}\,p\le\varepsilon$ half plane by half circle and pass to the limit when $\gamma\to\infty$. If we consider that integral along half circle (for positive ξ) tends to zero, when $\gamma\to\infty$ and use residual theory, we shall obtain:

$$x(\xi)=\mathrm{Re}\sum_j \mathrm{Res}\left[\frac{S}{m}\frac{e^{P\tau}}{p^2e^{P\tau}+\omega^2}e^{p\xi};p_j\right], \tag{6}$$

where summation is carried out to $\Phi(p)=p^2e^{p\tau}+\omega^2$ quasi-polynomial all p_j roots (that represent simple roots [2]), i.e. to equation solutions

$$p^2e^{p\tau}=-\omega^2\,. \tag{7}$$

On the basis of $\tau<<T$ condition and ε value we can consider that in the real part of the equation (7), any solution is negative $\mathrm{Re}\,p<0$.

If we locate $p_j=x_j+iy_j$ of the equation (7) according to their expansion (decrease of x_j abscissas) and equate the modulus in the equality (7), then we shall obtain:

$$e^{\tau x_j}(x_j^2+y_j^2)=\omega^2\,. \tag{8}$$

If now we consider that $\lim\limits_{j\to\infty}x_j=-\infty$, then from (8) we shall have:

$$\lim_{j\to\infty}e^{\tau x_j}y_j^2=\omega^2 \quad e^{\tau x_j}y_j^2=\omega^2+\varepsilon_{1j} \quad (\varepsilon_{1j}\to 0 \;\text{ when }\; j\to\infty\,). \tag{9}$$

Hence we shall obtain:

$$\lim_{\gamma\to\infty}\frac{y_j^2}{x_j^2}=\infty\,.$$

Since between the roots of the equation (7) we come across conjugated complex numbers, we can be satisfied by

$y_j > 0$ case discussion and consider that

$$\arg p_j = arctg\frac{y_j}{x_j} = \frac{\pi}{2} + \varepsilon_{2j} \quad (\varepsilon_{3j} \to 0 \quad \text{when} \quad j \to \infty). \tag{10}$$

Taking the logarithm of equality (7) we shall obtain

$$\tau p_j + 2\ln\left|p_j\right| + 2i\arg p_j = \ln\omega^2 + (2j+1)\pi i\,.$$

Whence we shall have

$$y_j = \frac{1}{\tau}[(2j+1)\pi - 2\arg p_j] = \frac{2\pi}{\tau}j + \varepsilon_{3j} \quad (\varepsilon_{3j} \to 0 \quad \text{when} \quad j \to \infty) \tag{11}$$

Regarding x_j, it will be determined from (9) condition

$$x_j = \frac{1}{\tau}[\ln\left(\omega^2 + \varepsilon_{1j}\right) - \ln y_j^2]$$

and, thus, asymptotic form of the p_j roots will have the following form

$$p_j \approx \frac{1}{\tau}\ln\frac{\omega^2\tau^2}{4\pi^2 j^2} + \frac{2\pi j}{\tau}i. \tag{12}$$

At the same time the error series in the given formula is $0\left(\frac{\ln_j}{j}\right)$.

The mentioned method enables us to estimate approximately large (so-called asymptotic) roots by quasi-polynomial modulus. As regards the non-asymptotic roots, we can find them by different approximate (for example Newton) methods.

If we consider that $\Phi(p)$ polynomial roots are simple roots and to calculate residues we shall use the formula

$$\operatorname{Re}s[\frac{\psi(z)}{\varphi(z)};a] = \frac{\psi(a)}{\varphi'(a)},$$

then from formula (6) we shall obtain

$$x(\xi) = \operatorname{Re}\left[\sum_j \frac{S}{mp_j\left(p_j\tau + 2\right)}e^{p_j\xi}\right]. \tag{13}$$

If we include value $\xi = t - \tau$, then the oscillator motion equation in the space time $t \in [\tau;\infty)$ will be described by the following equation

$$x(t) = \operatorname{Re}\sum_j \frac{S}{2m}\tau\left[\frac{1}{p_j\tau} - \frac{1}{p_j\tau + 2}\right]e^{p_j\xi}. \tag{14}$$

As we can see in (14), in the above obtained assumptions oscillator motion is a complex mechanical vibration and if we single out the main (dominant) vibration which has the largest amplitude (it corresponds to the smallest P_1 root with modulus), we can obtain $x(t) \approx \overset{0}{x}_1(t)$, where

$$\overset{0}{x}_1(t) = \frac{S}{2m}\tau\operatorname{Re}\left[\frac{1}{p_1\tau} - \frac{1}{p_1\tau + 2}\right]e^{P_1 t}, \qquad t \geq \tau\,.$$

Therewith,

$\left|\overset{0}{x}_1(t)\right| \le \frac{S}{m}\tau\frac{e^{x_1 t}}{|p_1\tau+2|}$ if $\tau x_1 \le -1$; $\left|\overset{0}{x}_1(t)\right| \le \frac{S}{m}\frac{e^{x_1 t}}{|p_1|}$, if $\tau x_1 \ge -1$ and, therefore, separated motion experiences damping in the form of the exponential function.

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