Mathematics

On the Almost Everywhere Convergence and Partial Sum’s Majorant of Series with Respect to Block-Orthonormal Systems

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ABSTRACT. In the present paper the behaviour of partial sum’s majorant of series with respect to block-orthonormal systems is considered and the estimates for partial sum’s majorant are established. © 2009 Bull. Georg. Natl. Acad. Sci.

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Let \( \{ \varphi_n \} \) be an orthonormal system from \( L^2(0,1) \). By \( \sigma(\{\varphi_n\}) \) we denote the set of all sequences \( \{a_n\} \) for which the series

\[
\sum_{n=1}^{\infty} a_n \varphi_n(x)
\]

converges almost everywhere on \((0,1)\).

K. Tandori [1] considered the set

\[ \sigma_\Omega = \bigcap_{|\varphi_n| \in \Omega} \sigma(\{\varphi_n\}), \]

where \( \Omega \) denotes the set of all orthonormal systems from \( L^2(0,1) \). He studied the quantity

\[
I(a_1, \ldots, a_n) = \sup_{|\varphi_n| \in \Omega} \left\{ \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_i \varphi_i(x) \right|^2 \right\} dx
\]

and by it for each sequence \( \{a_i\} \) he has determined

\[
\|a_i\| = \lim_{n \to \infty} I^{\frac{1}{2}}(a_1, \ldots, a_n).
\]

The quantity (3) always exists finite or infinite and it has all the properties of a norm.

Theorem (K. Tandori [1]). The class \( \sigma_\Omega \) coincides with the set of all sequences \( \{a_i\} \) for which the quantity
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(3) is finite.

If \( |a_1| \geq |a_2| \geq \ldots \), for norm (3) is fulfilled (see also [2,3]):

\[
c_i \sum_{n=1}^{\infty} a_n^2 \log^2 (n+1) \leq \left\| \{a_n\} \right\|^2 \leq c_1 \sum_{n=1}^{\infty} a_n^2 \log^2 (n+1).
\]

In the present paper we shall consider block-orthonormal systems and properties of partial sum's majorant of series with respect to block-orthonormal systems ([4,5]).

**Definition**([4]). Let \( \{N_k\} \) be an increasing sequence of natural numbers, \( \Delta_k = (N_k, N_{k+1}] \), \( k = 1, 2, \ldots \) and \( \{\phi_n\} \) be a system of functions from \( L^2(0,1) \). The system \( \{\phi_n\} \) will be called a \( \Delta_k \)-orthonormal system (\( \Delta_k \)-ONS) if:

1) \( \|\phi_n\| = 1, \ n = 1, 2, \ldots \);
2) \( (\phi_i, \phi_j) = 0 \), for \( i, j \in \Delta_k, \ i \neq j, \ k \geq 1 \).

For each \( \Delta_k \)-ONS \( \{\phi_n\} \) by \( \sigma(\{\phi_n\}, \Delta_k) \) we denote the set of all sequences \( \{a_n\} \), for which the corresponding series (1) converges almost everywhere on \( (0,1) \). \( \Omega(\Delta_k) \) denotes the set of all block-orthonormal systems from \( L^2(0,1) \). We denote:

\[
I_{\Delta_k}(a_1, \ldots, a_n) = \sup_{\{\phi_n\} \in \Omega(\Delta_k)} \int_0^1 \max_{i \leq n} \left| \sum_{i=1}^n a_i \phi_i(x) \right|^2 dx
\]

and

\[
\left\| \{a_n\}, \Delta_k \right\| = \lim_{k \to \infty} I_{\Delta_k}^{1/2}(a_1, \ldots, a_n).
\]

(4)

It is clear that \( \Omega \subset \Omega(\Delta_k) \), therefore

\[
I(a_1, \ldots, a_n) \leq I_{\Delta_k}(a_1, \ldots, a_n)
\]

and

\[
\left\| \{a_n\} \right\| \leq \left\| \{a_n\}, \Delta_k \right\|.
\]

**Lemma 1.** The following inequalities are fulfilled:

1) \( I_{\Delta_k}(a_1, \ldots, a_n) \geq \sum_{n=1}^{\infty} a_n^2 \);

2) \( I_{\Delta_k}(a_1, \ldots, a_n) \leq I_{\Delta_k}(a_1, \ldots, a_n, a_{n+1}) \);

(5)

3) \( I_{\Delta_k}^{1/2}(a_1 + b_1, \ldots, a_n + b_n) \leq I_{\Delta_k}^{1/2}(a_1, \ldots, a_n) + I_{\Delta_k}^{1/2}(b_1, \ldots, b_n) \);

4) \( I_{\Delta_k}(a_1, \ldots, a_n, b_1, \ldots, b_n) \geq I_{\Delta_k}(a_1, \ldots, a_n) + I_{\Delta_k} \left( 0, \ldots, 0, b_1, \ldots, b_n \right) \);

5) \( I_{\Delta_k}(a_1, \ldots, a_n) \leq I_{\Delta_k}(b_1, \ldots, b_n) \), if \( |a_i| \leq |b_i|, \ (1 \leq i \leq n) \).

We note that from (5) the existence of limit (4) (finite or infinite) follows

**Lemma 2.** Let for sequence \( \{a_n\} \) the condition

\[
\left\| \{a_n\}, \Delta_k \right\| < \infty
\]
be fulfilled. Then for every $\Delta_{x}$ -ONS $\{\varphi_n\}$ the corresponding series (1) converges by norm in $L^2(0,1)$.

Lemma 3. Let the condition

$$S^*_\varphi(\{a_i\},x) = \sup_{i \in N \cup \{\}} \sum_{n=1}^{\infty} a_n \varphi_n(x)$$

be fulfilled. Then for every sequence $\{a_i\}$ we have:

$$\sup_{\varphi \in \Omega(\Delta_x)} \left\| S^*_\varphi(\{a_i\},x) \right\|_2 = \left\| \{a_i\}, \Delta_x \right\|_2.$$

Let

$$\sigma_{i(\Delta_x)} = \bigcap_{\varphi \in \Omega(\Delta_x)} \sigma\{\varphi_n, \Delta_x\}.$$

The following theorems are fulfilled.

Theorem 1. $\sigma_{i(\Delta_x)}$ is the set of all sequences $\{a_i\}$, for which

$$\left\| S^*_\varphi(\{a_i\},x) \right\| \leq c,$$

where $\{\varphi_n\}$ is arbitrary $\Delta_{x}$ -ONS from $\Omega(\Delta_x)$. The constant $C$ is dependent on sequence $\{a_i\}$.

Theorem 2. If positive nondecreasing sequence $\{\omega(n)\}$ satisfies conditions

$$\sum_{i=1}^{\infty} \frac{1}{\omega(N_x)} < \infty \text{ and } \log^2 n = O(\omega(n)), \quad (n \to \infty),$$

then for arbitrary numbers $|a_1| \geq |a_2| \geq |a_3| \geq \ldots$ we have estimate

$$c_1 \sum_{n=1}^{\infty} a_n^2 \log^2 n \leq \sup_{\{\varphi_n\} \subset \Omega(\Delta_x)} \left\| S^*_\varphi(\{a_i\},x) \right\| \leq \sup_{\{\varphi_n\} \subset \Omega(\Delta_x)} \left\| S^*_\varphi(\{a_i\},x) \right\| \leq c_2 \sum_{n=1}^{\infty} a_n^2 \omega(n).$$

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