საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 3, №3, 2009
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 3, no. 3, 2009

Mathematics

On the Almost Everywhere Convergence and Partial Sum's Majorant of Series with Respect to Block-Orthonormal Systems

Givi Nadibaidze

I. Javakhishvili Tbilisi State University

(Presented by Academy Member E. Nadaraya)

ABSTRACT. In the present paper the behaviour of partial sum's majorant of series with respect to block-orthonormal systems is considered and the estimates for partial sum's majorant are established. *© 2009 Bull. Georg. Natl. Acad. Sci.*

***Key words**: block-orthonormal systems.*

Let $\{\varphi_n\}$ be an orthonormal system from $L^2(0,1)$. By $\sigma(\{\varphi_n\})$ we denote the set of all sequences $\{a_n\}$ for which the series

$$\sum_{n=1}^{\infty} a_n \varphi_n(x) \tag{1}$$

converges almost everywhere on (0,1).

K. Tandori [1] considered the set

$$\sigma_\Omega = \bigcap_{\{\varphi_n\}\in\Omega} \sigma(\{\varphi_n\}),$$

where Ω denotes the set of all orthonormal systems from $L^2(0,1)$. He studied the quantity

$$\mathrm{I}(a_1,\dots,a_n) = \sup_{\{\varphi_n\}\in\Omega} \int_0^1 \max_{1\le l\le n} \left| \sum_{i=1}^{l} a_i \varphi_i(x) \right|^2 dx \tag{2}$$

and by it for each sequence $\{a_i\}$ he has determined

$$\left\| \{a_i\} \right\| = \lim_{n\to\infty} I^{1/2}(a_1,\dots,a_n). \tag{3}$$

The quantity (3) always exists finite or infinite and it has all the properties of a norm.

Theorem (K. Tandori [1]). *The class σ_Ω coincides with the set of all sequences $\{a_i\}$, for which the quantity*

(3) *is finite.*

If $|a_1| \geq |a_2| \geq \ldots$, *for norm* (3) *is fulfilled* (see also [2,3]):

$$c_1 \sum_{n=1}^{\infty} a_n^2 \log_2^2(n+1) \leq \left\| \{a_i\} \right\|^2 \leq c_2 \sum_{n=1}^{\infty} a_n^2 \log_2^2(n+1).$$

In the present paper we shall consider block-orthonormal systems and properties of partial sum's majorant of series with respect to block-orthonormal systems ([4,5]).

Definition([4]). Let $\{N_k\}$ be an increasing sequence of natural numbers, $\Delta_k = (N_k, N_{k+1}]$, $(k = 1,2,\ldots)$ and $\{\varphi_n\}$ be a system of functions from $L^2(0,1)$. The system $\{\varphi_n\}$ will be called a Δ_k-orthonormal system (Δ_k-ONS) if:

1) $\|\varphi_n\|_2 = 1, \quad n = 1,2,\ldots$;

2) $(\varphi_i, \varphi_j) = 0$, for $i, j \in \Delta_k, \quad i \neq j, \quad k \geq 1$.

For each Δ_k-ONS $\{\varphi_n\}$ by $\sigma(\{\varphi_n\}, \Delta_k)$ we denote the set of all sequences $\{a_i\}$, for which the corresponding series (1) converges almost everywhere on (0,1). $\Omega(\Delta_k)$ denotes the set of all block-orthonormal systems from $L^2(0,1)$. We denote:

$$I_{\Delta_k}(a_1,\ldots,a_n) = \sup_{\{\varphi_n\} \in \Omega(\Delta_k)} \int_0^1 \max_{1 \leq l \leq n} \left| \sum_{i=1}^{l} a_i \varphi_i(x) \right|^2 dx$$

and

$$\left\| \{a_i\}, \Delta_k \right\| = \lim_{n \to \infty} I_{\Delta_k}^{1/2}(a_1,\ldots,a_n). \tag{4}$$

It is clear that $\Omega \subset \Omega(\Delta_k)$, therefore

$$I(a_1,\ldots a_n) \leq I_{\Delta_k}(a_1,\ldots,a_n)$$

and

$$\left\| \{a_i\} \right\| \leq \left\| \{a_i\}, \Delta_k \right\|.$$

Lemma 1. *The following inequalities are fulfilled:*

1) $I_{\Delta_k}(a_1,\ldots,a_n) \geq \sum_{i=1}^{n} a_i^2$;

2) $I_{\Delta_k}(a_1,\ldots a_n) \leq I_{\Delta_k}(a_1,\ldots,a_n,a_{n+1})$; (5)

3) $I_{\Delta_k}^{1/2}(a_1+b_1,\ldots,a_n+b_n) \leq I_{\Delta_k}^{1/2}(a_1,\ldots a_n) + I_{\Delta_k}^{1/2}(b_1,\ldots,b_n)$;

4) $I_{\Delta_k}(a_1,\ldots,a_n,b_1,\ldots b_m) \geq I_{\Delta_k}(a_1,\ldots,a_n) + I_{\Delta_k}(\underbrace{0,\ldots,0}_{n},b_1,\ldots,b_m)$;

5) $I_{\Delta_k}(a_1,\ldots,a_n) \leq I_{\Delta_k}(b_1,\ldots,b_n)$, *if* $|a_i| \leq |b_i|$, $(1 \leq i \leq n)$.

We note that from (5) the existence of limit (4) (finite or infinite) follows

Lemma 2. *Let for sequence* $\{a_i\}$ *the condition*

$$\left\| \{a_i\}, \Delta_k \right\| < \infty$$

be fulfilled. Then for every Δ_k -ONS $\{\varphi_n\}$ *the corresponding series* (1) *converges by norm in* $L^2(0,1)$.

Lemma 3. *Let the condition*

$$S_\varphi^*(\{a_i\},x) = \sup_{1\le N<\infty}\left|\sum_{i=1}^{N} a_i\varphi_i(x)\right|$$

be fulfilled.Then for every sequence $\{a_i\}$ *we have:*

$$\sup_{\varphi\in\Omega(\Delta_k)}\left\|S_\varphi^*(\{a_i\},x)\right\|_2 = \left\|\{a_i\},\Delta_k\right\|.$$

Let

$$\sigma_{\Omega(\Delta_k)} = \bigcap_{\{\varphi_n\}\in\Omega(\Delta_k)} \sigma(\{\varphi_n\},\Delta_k).$$

The following theorems are fulfilled.

Theorem 1. $\sigma_{\Omega(\Delta_k)}$ *is the set of all sequences* $\{a_i\}$, *for which*

$$\left\|S_\varphi^*(\{a_i\},x)\right\|_2 \le c,$$

where $\{\varphi_n\}$ *is arbitrary* Δ_k -ONS *from* $\Omega(\Delta_k)$. *The constant* C *is dependent on sequence* $\{a_i\}$.

Theorem 2. *If positive nondecreasing sequence* $\{\omega(n)\}$ *satisfies conditions*

$$\sum_{k=1}^{\infty}\frac{1}{\omega(N_k)}<\infty \text{ and } \log_2^2 n = O(\omega(n)),\ (n\to\infty),$$

then for arbitrary numbers $|a_1|\ge|a_2|\ge|a_3|\ge\ldots$ *we have estimate*

$$c_1\sum_{n=1}^{\infty}a_n^2\log_2^2 n \le \sup_{\{\varphi_n\}\in\Omega}\left\|S_\varphi^*(\{a_i\},x)\right\|_2 \le \sup_{\{\varphi_n\}\in\Omega(\Delta_k)}\left\|S_\varphi^*(\{a_i\},x)\right\|_2 \le c_2\sum_{n=1}^{\infty}a_n^2\omega(n).$$

The designated research has been fulfilled by financial support of the Georgian National Science Foundation, Grant № GNSF/ST08/3-393.

მათემატიკა

ბლოკებში ორთოგონალური მწკრივების თითქმის ყველგან კრებადობისა და კერძო ჯამთა მაჟორანტის შესახებ

გ. ნადიბაიძე

ი. ჯავახიშვილის სახ. თბილისის სახელმწიფო უნივერსიტეტი

(წარმოდგენილია აკადემიის წევრის ე. ნადარაიას მიერ)

ნაშრომში განხილულია ბლოკებში ორთონორმირებული სისტემების მიმართ მწკრივების კერძო ჯამთა მაჟორანტის თვისებები და დადგენილია კერძო ჯამთა მაჟორანტის გარკვეული შეფასება.

REFERENCES

1. *A. M. Olevski* (1975), Fourier Series with Respect to General Orthogonal Systems. Berlin.
2. *G. Alexits* (1961), Convergence Problems of Orthogonal Series. Publ. Hungar. Acad. Sci.
3. *B.S. Kashin and A.A Saakyan* (1989), Orthogonal Series. Engl. transl.: Translations of Mathematical Monographs, **75**, Amer. Math. Soc. Providnce, RI.
4. *V. F. Gaposhkin* (1990), Izv. VUZ, Ser. Mathem., **34**, 5.
5. *G. Nadibaidze* (1995), Georgian Math. J., **2,** 5: 517-534.

Received April, 2009