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Investigation of Metric Properties of Quality Criteria Space when Solving Multicriterion Optimization Problems

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ABSTRACT. The problem of the construction of an object functioning in the regime of optimum performance at the design stage is reduced to the solution of the problem of multicriterion optimization, where the quality criteria are chosen to be its most essential characteristics (parameters). At the same time in all methods of multicriterion optimization the vector quality criterion is considered basically in the linear Euclidean space. Actually, in most cases, the criterion space is non-Euclidean - it is curved. Therefore, such setting cannot give results adequately reflecting the processes running in real systems.

In order for the design system to really satisfy the optimality requirements the authors of the given paper offer an absolutely new approach to the solution of the problems of multicriterion optimization based on the definition of the quality criteria space and on finding an invariant corresponding to the distance between any two points of that space.

The idea of the study of the metric properties of the quality criteria space and their use in solving problems of optimization was offered in the work [1]. But that idea, due to its complexity, has not been completely realized until now. When solving such problems the quality criteria space was automatically identified with the Euclidean space with corresponding metrics. In the general case this couldn't give results adequately reflecting the processes occurring in real systems.

In the present paper metric properties of space criteria are studied for the first time, using as the main instrument the mathematical apparatus of tensor analysis, Riemannian geometry, differential equations in partial derivatives etc. Boundary problems relative to the components of the metric tensor of the *n*-dimensional space of the phenomenon states enabling to determine its metric properties are posed. The knowledge of the metric tensor furthers the objective appraisal of the phenomenon state and the definition of the optimal state. *© 2009 Bull. Georg. Natl. Acad. Sci.*

Key words: *multicriterion optimization, non-Euclidean space of criteria, metric properties, tensor analysis, Riemannian geometry, differential equations in partial derivatives, boundary problem.*

Introduction. In our daily life we very often estimate the states of different phenomena taking place around us. These phenomena can be: physical, biological, economic, social, etc. In the proposed paper the notion of a phenomenon is elementary, and it is not to be determined by more elementary notions.

The phenomena observed by us are characterized by definite parameters p^k (k=1,2,...,n). Such parameters can assume continuous or discrete values (a case of continuous parameters is considered below). Geometrically, in the n-dimensional space, to the change of these parameters corresponds some finite or infinite domain V_n- a domain of states. The form of the limiting hypersurface S of that domain completely depends on the essence of the observed phenomenon and on the limits of change of the characteristic parameters p^k (k=1,2,...,n).

The choice of the characteristic parameters p^k (k=1,2,...,n) represents a nontrivial problem; they can be determined only by highly qualified and experienced specialists in the corresponding field. For example, when estimating the colour, such parameters can be the intensities of separate constituent components of colour - red, green and yellow, and when estimating the quality of alloys - concentrations of separate constituents of the alloy, etc.

Henceforth we suppose that the system of characteristic parameters for that phenomenon is complete. The complete system of parameters p^k (k=1,2,...,n) should satisfy two conditions:

1. They unambiguously characterize the given phenomenon; to each separate state of the phenomenon correspond definite particular values of these parameters, and vice versa;

2. The parameters are independent, i.e. the following inequality takes place:

$$F\left(p^1, p^2, ..., p^n\right) \neq 0, \tag{1}$$

where F is a certain function.

In other respects, the choice of these parameters is random. If $\hat{p}^k$ (k=1,2,...,n) is another complete system of characteristic parameters, then between p^k and $\hat{p}^k$ (k=1,2,...,n) there should exist a functional unique connection

$$\hat{p}^k = f^k\left(p^1, p^2, ..., p^n\right) \quad (k=1,2,...,n). \tag{2}$$

Due to the unambiguousness of the functional dependence (2) the condition

$$\begin{vmatrix} \frac{\partial \hat{p}^1}{\partial p^1} & \frac{\partial \hat{p}^1}{\partial p^2} & \cdots & \frac{\partial \hat{p}^1}{\partial p^n} \\ \frac{\partial \hat{p}^2}{\partial p^1} & \frac{\partial \hat{p}^2}{\partial p^2} & \cdots & \frac{\partial \hat{p}^2}{\partial p^n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \hat{p}^n}{\partial p^1} & \frac{\partial \hat{p}^n}{\partial p^2} & \cdots & \frac{\partial \hat{p}^n}{\partial p^n} \end{vmatrix} \neq 0. \tag{3}$$

is true. The equality (2) is called a transformation of characteristic parameters. We shall give an example.

As is well known, any colour can be realized by mixing three rainbow colours, e.g. red, green and yellow. The corresponding parameters p^1, p^2 and p^3 indicate quotas of these colours in the total colour. If orange, blue and violet colours are used in the colour realization, then to the total colour will correspond other parameters $\hat{p}^1$, $\hat{p}^2$ and $\hat{p}^3$ unambiguously dependent on p^1, p^2 and p^3.

To one certain state of the observed phenomenon correspond certain values of the parameters p^k (k=1,2,...,n), i.e. one definite point in the domain V_n of the n-dimensional space of the state corresponds to it. Generally speaking, the n-dimensional space of the state is curved. Some line of such space represents a system of points, whose coordinates (k=1,2,...,n) depend on one scalar parameter t

$$p^k = p^k(t). \tag{4}$$

There can be an infinite number of lines between two points p_1^k and p_2^k (k=1,2,...,n). The space between these points is determined by the length of the shortest arch between p_1^k and p_2^k (k=1,2,...,n). The line to which corresponds the shortest arch is usually called a geodesic line [2].

The m-dimensional hypersurface of the n-dimensional ($m < n$) space represents a set of points, the coordinates of which are determined by the equalities

$$p^k = p^k\left(t^1, t^2, ..., t^m\right) \quad \left(2 \le m \le n-1\right), \tag{5}$$

where $t^1, t^2, ..., t^m$ are some parameters.

The study of all metric questions of the geometry of an n-dimensional space is reduced to the construction of the metric tensor of space [2-4].

1. Criteria of estimation of the phenomenon state. The state of phenomena taking place around us in itself is of no value; the estimation of the phenomenon state is a purely subjective notion. In the process of estimating the phenomenon state the following points are of decisive importance:

1. knowledge of the complete system of characteristic parameters;
2. knowledge of the criterion of the observed phenomenon state estimate.

It may occur that with the given values of the parameters p^k (k=1,2,...,n) the value of one parameter positively characterizes the phenomenon under consideration and the value of the other parameter - negatively (positivity and negativity are estimated from the viewpoint of the subject). For example:

1. The study of a student is estimated by marks in different subjects p^k (k=1,2,...,n). Here n is the number of subjects. Let us assume that the progress of a student in mathematics is estimated by the parameter p^1, and in history - by the parameter p^2. If p^1=5 and p^2=2 in the 5-point marks system, then it is evident that the value of the parameter p^1 positively characterizes the progress and the value of the parameter p^2 - negatively.

2. In a democratic society public matters are solved with the assistance of members of society. Among all the parameters characterizing the process of solving a certain public matter let us choose some: p^1 - the number of members of society taking part in solving the given problem; p^2 - the number of members of society having one vote each; p^3 - the number of members of society having two votes each, and so on. If the state of the phenomenon under consideration is characterized by the following values of these parameters: $p^1=N$, where N is the number of the population of the country whose age is over 16, $p^2=N$, $p^3=p^4=...=0$, then the value of the parameter p^1 positively characterizes the phenomenon under consideration, and the values of the parameter p^2, p^3,...- negatively, since, according to the value of the parameters p^2, p^3,... all members of society are equitable subjects independently of their abilities, merits and other characteristics. An analogous circumstance takes place in such phenomena as the protection of human rights, lawsuits, etc.

If a perfect objective method of estimating the phenomenon state is lacking, in most cases the estimate of the phenomenon state is carried out either by positive characteristics, or by negative ones; in one case positive estimates are taken into account, and in another - negative. The question why that occurs in such a way is beyond the scope of the present work. Actually such differentiation of views and lack of the criterion in estimating the phenomenon state cause a conflict between opposite sides.

The noted differentiation does not always take place. For example, in case of the colour analysis all the parameters p^1, p^2 and p^3 are of equal rights.

The objective estimate of the phenomenon state should concurrently take into account both positive and negative characteristics of that phenomenon.

The phenomenon state estimate can be carried out by introducing a numerical feature of the states' distinction, i.e. by introducing a distance between different states. In the Riemannian geometry [2] the distance between two points of the n-dimensional space is determined by the n-dimensional symmetric tensor of the second rank g_{ik}, $g_{ik}=g_{ki}$ (i,k =1,2,...,n). The components of that metric tensor represent functions of the parameters p^k (k=1,2,...,n).

Let p^k and p^k+dp^k (k=1,2,...,n) be two infinitesimal closely approximated states of the phenomenon under consideration. Here dp^k (k=1,2,...,n) are arbitrary infinitesimals. Then the distance dl between these states of the phenomenon under consideration equals [2-4]:

$$dl = \sqrt{e\, g_{ik}\, dp^i\, dp^k}\ . \tag{6}$$

(If the upper and the lower indices are repeated, then the summation from 1 to n is carried out by that index. In (6) the summation is fulfilled by the indices i and k, in particular, $g_{ik}dp^i dp^k = g_{11}dp^{1^2} + 2g_{12}dp^1 dp^2 + +2g_{13}dp^1 dp^3 + \cdots + g_{nn}dp^{n^2}$)

Here

$$e = \begin{cases} 1, & if \quad g_{ik} dp^i dp^k \geq 0, \\ -1, & if \quad g_{ik} dp^i dp^k < 0. \end{cases}$$

If the increments dp^k (k=1,2,...,n) of the parameters p^k (k=1,2,...,n) are carried out along a certain line $p^k=p^k(t)$ (k=1,2,...,n), then

$$dp^k = \dot{p}^k dt \quad (k = 1, 2, ..., n),$$

where $\dot{p}^k = \dfrac{dp^k}{dt}$ (k=1,2,...,n). Then (6) is as follows

$$dl = \sqrt{e\, g_{ik}\, \dot{p}^i \dot{p}^k}\; dt. \tag{7}$$

Besides, if a line (4) runs between two points

$$p_1^k = p^k(t_1) \text{ and } p_2^k = p^k(t_2) \; (k=1,2,...,n),$$

the finite length of the line arch between these points equals [2]:

$$l = \int_{t_1}^{t_2} \sqrt{e\, g_{ik}\, \dot{p}^i \dot{p}^k}\; dt. \tag{8}$$

If such line is geodesic, then l is minimal and thus represents the distance between the considered points, i.e. characterizes the degree of distinction between the two states p_1^k and p_2^k (k=1,2,...,n) of the phenomenon under consideration.

Equations of the geodesic line are determined especially from the principle of minimum of the functional (8). As is well known, the system of differential equations of the geodesic line is as follows [2-4]:

$$\ddot{p}^k + \Gamma_{ij}^k \dot{p}^i \dot{p}^j + \dot{p}^k \left(\frac{d^2 l}{dt^2} \Big/ \frac{dl}{dt} \right) = 0 \; (k=1,2,...,n). \tag{9}$$

Here

$$\ddot{p}^k = \frac{d^2 p^k}{dt^2} \quad (k = 1, 2, ..., n).$$

Besides

$$\Gamma_{ij}^k = \frac{1}{2} g^{kr} \left(\frac{\partial g_{ir}}{\partial p^j} + \frac{\partial g_{jr}}{\partial p^i} - \frac{\partial g_{ij}}{\partial p^r} \right) \quad (i, j, k = 1, 2, ..., n). \tag{10}$$

Γ_{ij}^k are denoted by the Kristoffel symbols of the second type [2-4]. From (10) it is evident that

$$\Gamma_{ij}^k = \Gamma_{ji}^k, \tag{11}$$

$$\Gamma_{ki}^k = \frac{\partial \ln \sqrt{|g|}}{\partial p^i} \quad (i, j, k = 1, 2, ..., n), \tag{12}$$

where g is a determinant composed of the elements of the metric tensor

$$g = \begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{vmatrix}. \qquad (13)$$

If we use the arch length t as a parameter, i.e. if $t = l$ then the system (9) will be simplified, and will be as follows:

$$\ddot{p}^k + \Gamma_{ij}^k \dot{p}^i \dot{p}^j = 0 \ \ (k=1,2,...,n). \qquad (14)$$

In that case, according to (7) we have

$$e g_{ik} \dot{p}^i \dot{p}^k = 1. \qquad (15)$$

That equality represents the first integral of the system of differential equations (14) can be directly proved [3,4].

Thus, if the metrics of the n-dimensional states space of the considered phenomenon is known, then by solving the system of differential equations (14) in the corresponding boundary conditions

$$p_0^k = p^k(0), \quad p_1^k = p^k(l_0) \quad (k = 1,2,...,n)$$

we shall determine the geodesic line $p^k = p^k(l)$ running through the points p_0^k and p_1^k (k=1,2,...,n) (here l is counted out from the first point p_0^k (k=1,2,...,n)). Substituting the functions $p^k = p^k(l)$ in (15), and assuming that $l = l_0$ we shall get the equation relative to l_0, by the solution of which the arch length of the geodesic line between the points p_0^k and p_1^k is determined, i.e. the measure of discrepancy between the two states p_0^k and p_1^k of the considered phenomenon is determined.

In some cases among the states of the observed phenomenon, in the view of the researcher, there exists the best state p_0^k (k=1,2,...,n). The best state (we shall designate it "ideal") can be practically realizable and practically unrealizable (utopian). In the first case the point $M_0\left(p_0^1, p_0^2, ..., p_0^n\right)$ belongs to the domain V_n, i.e. $M_0\left(p_0^1, p_0^2, ..., p_0^n\right) \in V_n$, and in the second case it lies out of the domain V_n, i.e. $M_0\left(p_0^1, p_0^2, ..., p_0^n\right) \notin V_n$. For example:

1. In the process of the student's study the best state is the state under which all p_0^k (k=1,2,...,n) equal five, i.e. p_0^k =5 (k=1,2,...,n). That best state is practically realizable, i.e. $M_0(5,5,...,5) \in V_n$;

2. Communism, being one of the best states of the social phenomenon (social order), is a utopian state. No social order can satisfy all desires of the people, since life itself is a tireless competition among people. In that case the point M_0, corresponding to the best state, lies out of the domain V_n.

Criteria of the state estimate. *Estimate of the given state* p^k (k=1,2,...,n) *of the phenomenon under consideration is determined by the distance between the "ideal" and the given states, i.e. by the distance in the n-dimensional states space between the two points* p_0^k *and* p^k (k=1,2,...,n) *corresponding to both the "ideal" state and the given state.*

In the case of the phenomena when the estimation of the phenomenon state is not required as it is, for example, for colour, the space (distinction) between two arbitrary states of the observed phenomenon can be determined on the basis of the above-cited method.

The estimation of the phenomenon state allows to choose from the set of several states $p_1^k, p_2^k, ..., p_N^k$ (k=1,2,...,n) the best one among that great number. To the best state among all the states corresponds the least distance to the "ideal" state p_0^k (k=1,2,...,n), i.e.

$$l_0 = \min\left(l_{0,1}, l_{0,2}, ..., l_{0,N}\right). \qquad (16)$$

Remark 1. The assessment criterion of the state can be formed proceeding from the worst state of the phenomenon under consideration. Here the parameters p_0^k (k=1,2,...,n) characterize the worst state of the phenomenon.

Remark 2. When forming the characteristic parameters p^k (k=1,2,...,n) of the phenomenon under consideration some parameters, especially those having integral character, can be formed while using the above indicated criterion of the phenomenon state estimate.

2. The system of differential equations relative to the components of the metric tensor. It was noted that the method of the state estimation of some phenomenon is completely based on using the metric tensor of space of the phenomenon states. Here we shall try to give the most general method of forming a system of differential equations and corresponding boundary conditions, by the solution of which separate components of the metric tensor in different particular cases can be determined. The essence of that method is taken from Einstein's relativistic theory of the gravitational field [6]. At the same time, we shall try to prove axiomatically the problem posed in the given section.

Axiom 1. The system of differential equations relative to the components of the metric tensor of space of the states of the phenomenon under consideration should be of the second order relative to the derivatives of these components by the variables p^k (k=1,2,...,n) .

Axiom 2. The system of differential equations relative to the components of the metric tensor of the space should be invariant relative to the choice of the complete characteristic parameters p^k (k=1,2,...,n) of the phenomenon under consideration, i.e. it should be invariant relative to the transformation (2).

Axiom 3. The solutions of the system of differential equations relative to the components of the metric tensor of space of the phenomenon states represent minimizing functions of some invariant (relative to the transformation (2)) functional $\int\limits_{V_n} L dp$, i.e.

$$\delta \int\limits_{V_n} L dp = 0, \tag{17}$$

where $L = L\left(g_{11}, g_{12},..., g_{nn}; \frac{\partial g_{11}}{\partial p^k}, \frac{\partial g_{12}}{\partial p^k},..., \frac{\partial g_{nn}}{\partial p^k} \right)$ is some invariant function of the components of the metric tensor and their partial derivatives of the first order by the variables p^k (k=1,2,...,n), and dp is the element of the volume of the n-dimensional space of the phenomenon states.

From these three axioms it is evident that the procedure of forming the system of differential equations of the indicated type is reduced to forming the function L and the elementary volume dp of the n-dimensional space under consideration. The formation of L and dp, for its part, is closely connected with the main issues of the Riemannian geometry. We shall quote the statement of such problems without proofs within the framework of that work.

The curvature tensor (the Riemannian tensor) of the n-dimensional curved space with the metric tensor g_{ij} (k=1,2,...,n) is as follows [2-4]:

$$R_{ijl}^{k} = \frac{\partial \Gamma_{il}^{k}}{\partial p^j} - \frac{\partial \Gamma_{ij}^{k}}{\partial p^l} + \Gamma_{il}^{p}\Gamma_{pj}^{k} - \Gamma_{ij}^{p}\Gamma_{pl}^{k} \qquad (i, j, k, l = 1, 2, ..., n). \tag{18}$$

From it we can form a lower order tensor - Levy-Chivit tensor:

$$R_{ij} = R_{ijk}^{k} = \frac{\partial \Gamma_{ik}^{k}}{\partial p^j} - \frac{\partial \Gamma_{ij}^{k}}{\partial p^k} + \Gamma_{ik}^{p}\Gamma_{pj}^{k} - \Gamma_{ij}^{p}\Gamma_{pk}^{k} \quad (i, j = 1, 2, ..., n) \tag{19}$$

and a scalar curvature:

$$R = g^{ij} R_{ijk}^{k} = g^{ij}\left(\frac{\partial \Gamma_{ik}^{k}}{\partial p^j} - \frac{\partial \Gamma_{ij}^{k}}{\partial p^k} + \Gamma_{ik}^{p}\Gamma_{pj}^{k} - \Gamma_{ij}^{p}\Gamma_{pk}^{k} \right). \tag{20}$$

From the expression (20) it is evident that it contains the components of the metric tensor and their derivatives of the first and second orders, but it contains the derivatives of the second order as a summand of a divergence type. Taking into account that for the n-dimensional space under consideration other invariant values of L type don't exist, and the circumstance that in the expression L the summand of a divergence type does not affect the formation of the Euler-Lagrange differential equations [5], we shall determine L by the equality:

$$L=R. \tag{21}$$

Let us introduce antisymmetric symbols $\varepsilon_{i_1 i_2 \dots i_n}$ on all indices [4] for determining the volume element dp of the n-dimensional space of the phenomenon state. If two arbitrary indices are equal, then the corresponding symbol equals zero. We equate the symbol $\varepsilon_{1,2,\dots,n}$ with the unit $\varepsilon_{1,2,\dots,n}=1$, then the nonzero elements are determined by the following equalities:

$$\varepsilon_{i_1 i_2 \dots i_n} = \begin{cases} 1, & if \quad p(i_1, i_2, \dots, i_n) - even\ number, \\ -1, & if \quad p(i_1, i_2, \dots, i_n) - odd\ number. \end{cases}$$

Here $p(i_1, i_2, \dots, i_n)$ is the number of permutations necessary for reducing the sequence of numbers $i_1, i_2, \dots, i_n$ to the sequence 1,2,...,n. By the definition of the determinant formed from the number a^{ik} (i,k = 1,2,...,n) it is evident that

$$\varepsilon_{i_1 i_2 \dots i_n} a^{i_1 k_1} a^{i_2 k_2} \cdots a^{i_n k_n} = (-1)^{p(k_1, k_2, \dots, k_n)} \begin{vmatrix} a^{1,1} & a^{1,2} & \dots & a^{1,n} \\ a^{2,1} & a^{2,2} & \dots & a^{2,n} \\ \dots & \dots & \dots & \dots \\ a^{n,1} & a^{n,2} & \dots & a^{n,n} \end{vmatrix}, \tag{22}$$

when all the numbers $k_1, k_2, \dots, k_n$ differ from each other, and otherwise the given expression equals zero.

The symbols $\varepsilon_{i_1 i_2 \dots i_n}$ don't constitute a tensor. The tensor is a set of values [4]:

$$e_{i_1 i_2 \dots i_n} = \sqrt{|g|}\ \varepsilon_{i_1 i_2 \dots i_n}. \tag{23}$$

Let us introduce the following n linear-independent vectors

$$dp^k_{1\bullet},\ dp^k_{2\bullet},\ \dots,\ dp^k_{n\bullet}.$$

The lower indices indicate vector numbers. They represent numbers of the vectors' components, therefore they are designated by dots.

According to (22) it is evident that

$$dp = e_{i_1 i_2 \dots i_n} dp^{i_1}_{1\bullet}\ dp^{i_2}_{2\bullet} \dots dp^{i_n}_{n\bullet} = \sqrt{|g|} \begin{vmatrix} dp^1_{1\bullet} & dp^2_{1\bullet} & \dots & dp^n_{1\bullet} \\ dp^1_{2\bullet} & dp^2_{2\bullet} & \dots & dp^n_{2\bullet} \\ \dots & \dots & \dots & \dots \\ dp^1_{n\bullet} & dp^2_{n\bullet} & \dots & dp^n_{n\bullet} \end{vmatrix}. \tag{24}$$

From the last expression it is evident that it represents an invariant element of the volume of the n-dimensional space of the phenomenon states under consideration [4]. When using this equality, we can carry out the reduction of the volumetric integral from the value of the divergence type to the surface integral.

Let us consider the integral

$$\int_{V_n} \frac{1}{\sqrt{|g|}} \frac{\partial\left(\sqrt{|g|}A^k\right)}{\partial p^k} dp = \int_{V_n} \frac{\partial\left(\sqrt{|g|}A^k\right)}{\partial p^k} \varepsilon_{i_1 i_2 \ldots i_n} dp_{1\bullet}^{i_1}\, dp_{2\bullet}^{i_2} \ldots dp_{n\bullet}^{i_n}, \tag{25}$$

where A^k (k=1,2,...,n) is a certain vector depending on the variables p^k (k=1,2,...,n). For calculating the first summand in the right part of that equality we assume that

$$dp_{1\bullet}^1 = dp^1\,, \quad dp_{1\bullet}^2 = dp_{1\bullet}^3 = \cdots = dp_{1\bullet}^n = 0\,,$$

and $dp_{2\bullet}^{i_2}, dp_{3\bullet}^{i_3}, \ldots, dp_{n\bullet}^{i_n}$ are arbitrary infinitely small vectors, whose values on the hypersurface S of the n–1 dimension, bounding the domain V_n we designate by $dp_{2\bullet S}^{i_2}, dp_{3\bullet S}^{i_3}, \ldots, dp_{n\bullet S}^{i_n}$, i.e. $dp_{2\bullet}^{i_2} = dp_{2\bullet S}^{i_2}$, $dp_{3\bullet}^{i_3} = dp_{3\bullet S}^{i_3}, \ldots, dp_{n\bullet}^{i_n} = dp_{n\bullet S}^{i_n}$ on S. At that, the vectors $dp_{2\bullet S}^{i_2}, dp_{3\bullet S}^{i_3}, \ldots, dp_{n\bullet S}^{i_n}$ completely lie on S (the tangent vectors to S). Then the first summand in the right part of the equality (25) will be as follows:

$$\int_{V_n} \frac{\partial\left(\sqrt{|g|}A^1\right)}{\partial p^1} e_{1,i_2 \ldots i_n} dp^1\; dp_{2\bullet}^{i_2} \ldots dp_{n\bullet}^{i_n} = \int_S A^1 \sqrt{|g|}\, \varepsilon_{1,i_2 \ldots i_n} dp_{2\bullet S}^{i_2}\, dp_{3\bullet S}^{i_3} \ldots dp_{n\bullet S}^{i_n}\,.$$

By analogy, if we assume that $dp_{1\bullet}^1 = 0$, $dp_{1\bullet}^2 = dp^2$, $dp_{1\bullet}^3 = dp_{1\bullet}^4 = \cdots = dp_{1\bullet}^n = 0$, then for the second summand in the right part of the equality (25) we get:

$$\int_{V_n} \frac{\partial\left(\sqrt{|g|}A^2\right)}{\partial p^2} \varepsilon_{2,i_2 \ldots i_n} dp^2 dp_{2\bullet}^{i_2} dp_{3\bullet}^{i_3} \ldots dp_{n\bullet}^{i_n} = \int_S A^2 \sqrt{|g|}\, \varepsilon_{2,i_2 \ldots i_n} dp_{2\bullet S}^{i_2}\, dp_{3\bullet S}^{i_3} \ldots dp_{n\bullet S}^{i_n}\,.$$

Proceeding with this process, the equality (25) will assume the following form:

$$\int_{V_n} \frac{1}{\sqrt{|g|}} \frac{\partial\left(\sqrt{|g|}A^k\right)}{\partial p^k} dp = \int_S A^k \sqrt{|g|}\, \varepsilon_{k i_2 \ldots i_n} dp_{2\bullet S}^{i_2}\, dp_{3\bullet S}^{i_3} \ldots dp_{n\bullet S}^{i_n}\,. \tag{26}$$

The last equation establishes the connection between volumetric and surface integrals.

Let us introduce the designations:

$$dS_k = \sqrt{|g|}\, \varepsilon_{k i_2 \ldots i_n} dp_{2\bullet S}^{i_2}\, dp_{3\bullet S}^{i_3} \ldots dp_{n\bullet S}^{i_n} \quad \left(k = 1,2,\ldots,n\right). \tag{27}$$

Taking into account that $dp_{2\bullet S}^{i_2}\, dp_{3\bullet S}^{i_3} \ldots dp_{n\bullet S}^{i_n}$ is a contravariant tensor of the n–1 order, then it is evident that according to (23) dS_k (k=1,2,...,n) represents a covariant vector. It is called a dual vector of the tensor $dp_{2\bullet S}^{i_2}\, dp_{3\bullet S}^{i_3} \ldots dp_{n\bullet S}^{i_n}$. From the vector structure dS_k (k=1,2,...,n) it is evident that it is orthogonal to all vectors $dp_{2\bullet S}^{i_2}$, $dp_{3\bullet S}^{i_3}, \ldots, dp_{n\bullet S}^{i_n}$, i.e.

$$dp_{2\bullet S}^k\, dS_k = dp_{3\bullet S}^k\, dS_k = \cdots = dp_{n\bullet S}^k\, dS_k = 0\,.$$

In these expressions we have:

$$\int_{V_n} \frac{1}{\sqrt{|g|}} \frac{\partial\left(\sqrt{|g|}A^k\right)}{\partial p^k} dp = \int_S A^k dS_k\,. \tag{28}$$

Here we don't give a detailed elucidation of these questions. Those wishing can refer to [2,4].

For the aim of extracting the summand of the divergence type from the expression R, let us transform and rewrite it as follows:

$$R = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial p^k}\left[\sqrt{|g|}\,g^{ij}\left(\delta^k_j\Gamma^p_{ip} - \Gamma^k_{ij}\right)\right] - \frac{1}{\sqrt{|g|}}\left(\delta^k_j\Gamma^p_{ip} - \Gamma^k_{ij}\right)\frac{\partial\left(\sqrt{|g|}\,g^{ij}\right)}{\partial p^k} + g^{ij}\left(\Gamma^p_{ik}\Gamma^k_{pj} - \Gamma^p_{ij}\Gamma^k_{pk}\right). \quad (29)$$

Here δ^k_i is the Kroneker symbol (the mixed tensor of the second order):

$$\delta^k_i = \begin{cases} 1, & at \quad i = k, \\ 0, & at \quad i \neq k. \end{cases}$$

From this expression of the scalar curvature R it is obvious that the first summand in the right part, looking like a divergence, contains the second derivatives of the components of the metric tensor g_{ik} (k=1,2,...,n) from the variables p^k (k=1,2,...,n), and the remaining summands contain only the first derivatives.

For convenient recording we shall use the following designations:

$$A^k = g^{ij}\left(\delta^k_j\Gamma^p_{ip} - \Gamma^k_{ij}\right),$$

$$L^* = -\frac{1}{\sqrt{|g|}}\left(\delta^k_j\Gamma^p_{ip} - \Gamma^k_{ij}\right)\frac{\partial}{\partial p^k}\left(\sqrt{|g|}\,g^{ij}\right) + g^{ij}\left(\Gamma^p_{ik}\Gamma^k_{pj} - \Gamma^p_{ij}\Gamma^k_{pk}\right). \quad (30)$$

Then, according to (20), we have

$$L = \frac{1}{\sqrt{|g|}}\frac{\partial\left(\sqrt{|g|}\,A^k\right)}{\partial p^k} + L^*. \quad (31)$$

So, according to (17) and (24), we should minimize the following functional:

$$I = \int\limits_{V_n}\frac{1}{\sqrt{|g|}}\frac{\partial\sqrt{|g|}A^k}{\partial p^k}e_{i_1 i_2 \dots i_n}dp^{i_1}_{1\bullet}dp^{i_2}_{2\bullet}\dots dp^{i_n}_{n\bullet} + \int\limits_{V_n}L^* e_{i_1 i_2 \dots i_n}dp^{i_1}_{1\bullet}dp^{i_2}_{2\bullet}\dots dp^{i_n}_{n\bullet}. \quad (32)$$

From the condition δI=0 we have

$$\int\limits_{V_n}\frac{\partial}{\partial p^k}\left[\delta\left(\sqrt{|g|}A^k\right) + \frac{\partial\left(\sqrt{|g|}L^*\right)}{\partial\left(\dfrac{\partial g_{\alpha\beta}}{\partial p^k}\right)}\delta g_{\alpha\beta}\right]\varepsilon_{i_1 i_2 \dots i_n}dp^{i_1}_{1\bullet}dp^{i_2}_{2\bullet}\dots dp^{i_n}_{n\bullet} +$$

$$+\int\limits_{V_n}\left[\frac{\partial\left(\sqrt{|g|}L^*\right)}{\partial g_{\alpha\beta}} - \frac{\partial}{\partial p^k}\left(\frac{\partial\left(\sqrt{|g|}L^*\right)}{\partial\left(\dfrac{\partial g_{\alpha\beta}}{\partial p^k}\right)}\right)\right]\delta g_{\alpha\beta}\varepsilon_{i_1 i_2 \dots i_n}dp^{i_1}_{1\bullet}dp^{i_2}_{2\bullet}\dots dp^{i_n}_{n\bullet} = 0. \quad (33)$$

Here the summation from 1 to n is carried out by the indices k, α, β. The integrand of the first summand in the left part of that equality looks like a divergence, therefore, according to (28) the latter equality will be as follows:

$$\int_S \left[\delta\left(\sqrt{|g|}A^k\right) + \sqrt{|g|}\frac{\partial L^*}{\partial\left(\frac{\partial g_{\alpha\beta}}{\partial p^k}\right)}\delta g_{\alpha\beta} \right] dS_k +$$

$$+\int_{V_n} \left[\frac{\partial\left(\sqrt{|g|}L^*\right)}{\partial g_{\alpha\beta}} - \frac{\partial}{\partial p^k}\left(\frac{\partial\left(\sqrt{|g|}L^*\right)}{\partial\left(\frac{\partial g_{\alpha\beta}}{\partial p^k}\right)} \right) \right] \delta g_{\alpha\beta}\varepsilon_{i_1 i_2 \ldots i_n} dp_{1\cdot}^{i_1} dp_{2\cdot}^{i_2} \ldots dp_{n\cdot}^{i_n} = 0 . \tag{34}$$

The second integral in the left part of this equality covers the *n*-dimensional domain V_n, and the first – the bounding domain V_n of the hypersurface *n*–1 of the dimension *S*, therefore

$$\int_{V_n} \left[\frac{\partial\left(\sqrt{|g|}L^*\right)}{\partial g_{\alpha\beta}} - \frac{\partial}{\partial p^k}\left(\frac{\partial\left(\sqrt{|g|}L^*\right)}{\partial\left(\frac{\partial g_{\alpha\beta}}{\partial p^k}\right)} \right) \right] \delta g_{\alpha\beta}\varepsilon_{i_1 i_2 \ldots i_n} dp_{1\cdot}^{i_1} dp_{2\cdot}^{i_2} \ldots dp_{n\cdot}^{i_n} = 0 ,$$

$$\int_S \left[\delta\left(\sqrt{|g|}A^k\right) + \sqrt{|g|}\frac{\partial L^*}{\partial\left(\frac{\partial g_{\alpha\beta}}{\partial p^k}\right)}\delta g_{\alpha\beta} \right] dS_k = 0 . \tag{35}$$

From the first equation of that system we have:

$$\frac{\partial}{\partial p^k}\left(\frac{\partial\left(\sqrt{|g|}L^*\right)}{\partial\left(\frac{\partial g_{ij}}{\partial p^k}\right)} \right) - \frac{\partial\left(\sqrt{|g|}L^*\right)}{\partial g_{ij}} = 0 \quad \left(i, j = 1,2,\ldots,n\right) . \tag{36}$$

If here we substitute the value L^* from (30), then after some transformation we shall finally get

$$R_{ij} - \frac{1}{2} g_{ij} R = 0 \ \left(i, j = 1,2,\ldots,n\right) . \tag{37}$$

The expression (37) represents a system of differential equations of the second order relative to the required components of the metric tensor g_{ij} $\left(i, j = 1,2,\ldots,n\right)$. The system can be simplified. To this end we shall multiply it by g^{ij} and sum over indices *i* and *j* from one to *n*, and we shall get *R*=0. So (37) will become:

$$R_{ij} = 0 \ \left(i, j = 1,2,\ldots,n\right) \text{ in } V. \tag{38}$$

Thus, (37) and (38) are equivalent systems of differential equations relative to g_{ij} $\left(i, j = 1,2,\ldots,n\right)$.

The second equality of the system (35) contains various boundary conditions, and the components of the metric tensor on the hypersurface S should satisfy them. That equality requires special analysis.

3. Boundary conditions relative to the metric tensor components. In the left part of the second equality of the system (35) the integrand contains the arbitrary functions $\delta g_{\alpha\beta}$ $(\alpha,\beta=1,2,...,n)$ and their partial derivatives relative to the variables p^k (k=1,2,...,n) defined on the hypersurface S. The derivatives $\frac{\partial(\delta g_{\alpha\beta})}{\partial p^k}$ $(\alpha,\beta=1,2,...,n)$ partially depend on $\delta(g_{\alpha\beta})$ $(\alpha,\beta=1,2,...,n)$, and partially on other arbitrary functions. In order to explain this functional dependence we shall introduce internal coordinates $q^1,q^2,\ldots,q^{n-1}$ on the hyperspace S. Corresponding coordinate lines belong to S, analogously to the case of the Euclidean space of three dimensions, where on the spherical surface are used geographic coordinates (internal coordinates) ϑ and φ, which with the Cartesian coordinates are connected by the equalities:

$$\begin{aligned} &x = R\sin\vartheta\cos\varphi, \quad y = R\sin\vartheta\sin\varphi, \\ &z = R\cos\vartheta, \quad 0\le\vartheta\le\pi, \quad 0\le\varphi\le 2\pi. \end{aligned} \tag{39}$$

Here R is a spherical surface radius. Meridians and parallels are coordinate lines.

Let us represent the hypersurface S by the parametric equations

$$p^k = p^k\left(q^1,q^2,\ldots,q^{n-1}\right) \quad (k=1,2,...,n). \tag{40}$$

It is evident that

$$\tau^k_{l\cdot} = \frac{\partial p^k}{\partial q^{l\cdot}} \quad (k=1,2,...,n; \quad l=1,2,...,n-1) \tag{41}$$

for every fixed value of the index l are components of the tangent vector $\vec{\tau}_{l\cdot}$ of the l-th coordinate line lying on S. Since the functions $\delta(g_{\alpha\beta})$ $(\alpha,\beta=1,2,...,n)$ are defined on S, it is impossible to define derivatives of those functions by the directions issuing from S. By such values of the functions under consideration derivatives by tangent directions of the surface S can be determined, in particular, by the directions defined by the equalities of (41)

$$\frac{\partial(\delta g_{\alpha\beta})}{\partial \tau_{l\cdot}} \quad (\alpha,\beta=1,2,...,n; \quad l=1,2,...,n-1),$$

i.e.

$$\frac{\partial(\delta g_{\alpha\beta})}{\partial p^k}\cdot\frac{\partial p^k}{\partial q^{l\cdot}} = \frac{\partial(\delta g_{\alpha\beta})}{\partial \tau_{l\cdot}} \quad (\alpha,\beta=1,2,...,n; \quad l=1,2,...,n-1). \tag{42}$$

For every fixed value of the indices α and β (42) represents a linear system consisting of n–1 equations relative to unknown $\frac{\partial(\delta g_{\alpha\beta})}{\partial p^k}$ (k=1,2,...,n), the number of which equals n. We assume that internal coordinate lines are nowhere tangent with each other, i.e. the matrix rank of the system (42), $\left\|\frac{\partial p^k}{\partial q^{l\cdot}}\right\|$, is equal to n–1. In such conditions the solution of the system (42) relative to $\frac{\partial(\delta g_{\alpha\beta})}{\partial p^k}$ (k=1,2,...,n) for each fixed value of the indices α, β is ambiguously determined. For a unique determination of the sought unknowns we should also know the derivative of the function $\delta g_{\alpha\beta}$ by some direction deriving from the surface *S*. If l^k (k=1,2,...,n) is a unit contravariant vector $\left(g_{ij}l^i l^j = 1\right)$ not lying on *S* (issuing from *S*), the condition

$$\begin{vmatrix} l^1 & l^2 & \dots & l^n \\ \dfrac{\partial p^1}{\partial q^1} & \dfrac{\partial p^2}{\partial q^1} & \dots & \dfrac{\partial p^n}{\partial q^1} \\ \dots & \dots & \dots & \dots \\ \dfrac{\partial p^n}{\partial q^{n-1}} & \dfrac{\partial p^n}{\partial q^{n-1}} & \dots & \dfrac{\partial p^n}{\partial q^{n-1}} \end{vmatrix} \neq 0 . \tag{43}$$

should take place. The derivative in the line of this vector equals

$$l^k \frac{\partial\left(\delta g_{\alpha\beta}\right)}{\partial p^k} = \frac{\partial\left(\delta g_{\alpha\beta}\right)}{\partial l}. \tag{44}$$

The right part of this equality is independent of the values $\delta g_{\alpha\beta}$ on *S* and represents an arbitrary function. The solution of the system consisting of (42) and (44), taking into account (43), uniquely defines the derivatives $\dfrac{\partial\left(\delta g_{\alpha\beta}\right)}{\partial p^k}$ $\left(\alpha,\beta,k = 1,2,...,n\right)$ depending on the arbitrary functions $\delta g_{\alpha\beta}$ and $\dfrac{\partial\left(\delta g_{\alpha\beta}\right)}{\partial l}$ $\left(\alpha,\beta = 1,2,...,n\right)$, the number of which equals $n(n+1)$.

In the first place let us consider the case when the arbitrary functions $\delta g_{\alpha\beta}$ and $\dfrac{\partial\left(\delta g_{\alpha\beta}\right)}{\partial l}$ $\left(\alpha,\beta = 1,2,...,n\right)$ are not bounded by additional conditions. In this case the second equality of the system (35) is satisfied automatically if

$$\begin{cases} g_{\alpha\beta}\Big|_S \ \textit{the given function}, \\ \dfrac{\partial g_{\alpha\beta}}{\partial l}\Big|_S \ \textit{the given function}, \ \left(\alpha,\beta = 1,2,...,n\right). \end{cases} \tag{45}$$

Really, at that

$$\delta g_{\alpha\beta} = 0\,, \quad \frac{\partial\left(\delta g_{\alpha\beta}\right)}{\partial l} = 0 \ \text{ on } S \ \left(\alpha,\beta = 1,2,...,n\right) \tag{46}$$

and the homogeneous system implying (42) and (44), in accordance with the condition (43) has only a trivial solution

$$\frac{\partial\left(\delta g_{\alpha\beta}\right)}{\partial p^k} = 0 \ \text{ on } S \ \left(\alpha,\beta,k = 1,2,...,n\right). \tag{47}$$

Therefore, from (46) and (47) the validity of the second equality of the system (35) is evident.

(38) and (45) form the following boundary problem relative to the components of the metric tensor $g_{\alpha\beta}$ $(\alpha, \beta = 1, 2, ..., n)$: to find in the V domain a regular solution (in the sense of the existence of the required order derivatives) of the system of differential equations (38) relative to unknown $g_{\alpha\beta}$ $(\alpha, \beta = 1, 2, ..., n)$, satisfying boundary conditions (45).

In accordance with the results of the theory of differential equations of mathematical physics [6] the problem in the estimation theory most likely can be used in the case of open surfaces S (in case of infinite domains).

Depending on concrete additional conditions relative to the arbitrary functions

$$\delta g_{\alpha\beta} \text{ and } \frac{\partial(\delta g_{\alpha\beta})}{\partial l} \quad (\alpha, \beta = 1, 2, ..., n), \tag{48}$$

from the second equation of the system (35) we shall get corresponding boundary conditions relative to the components of the metric tensor $g_{\alpha\beta}$ $(\alpha, \beta = 1, 2, ..., n)$. Here we shall consider one of such additional conditions, in particular, we shall assume that the arbitrary functions (48) satisfy the following nonholonomic additional conditions:

$$\left(\frac{1}{2} g^{pk} g^{\alpha\beta} - g^{k\alpha} g^{p\beta}\right)\left(\delta g_{\alpha\beta}\right)_{,p} = 0 \quad (k = 1, 2, ..., n), \tag{49}$$

where

$$\left(\delta g_{\alpha\beta}\right)_{,p} = \frac{\partial(\delta g_{\alpha\beta})}{\partial p^p} - \Gamma^l_{\alpha\beta}\left(\delta g_{l\beta}\right) - \Gamma^l_{\beta p}\left(\delta g_{l\alpha}\right) \quad (\alpha, \beta, p = 1, 2, ..., n) \tag{50}$$

represent a covariant derivative of the tensor $\delta g_{\alpha\beta}$. $\left(\delta g_{\alpha\beta}\right)_{,p}$ is a covariant tensor of the third order. The conditions (49) do not limit the degree of freedom of the arbitrary functions $\delta g_{\alpha\beta}$ $(\alpha, \beta = 1, 2, ..., n)$ since the total amount of the arbitrary functions (48) equals $n(n+1)$, and the number of the boundary conditions (49) equals n, at that $n(n+1)>n$. The degree of freedom of the arbitrary functions (48) $n(n+1)-n=n^2$ is not less than $\frac{n(n+1)}{2}$.

Moreover, the conditions (49) are invariant, they keep their image in all systems of complete parameters p^k (k=1,2,...,n), i.e. they change their image when transforming

$$p'^k = p'^k\left(p^1, p^2, \dots, p^n\right) \quad (k=1,2,...,n). \tag{51}$$

This is evident from the fact that $\frac{1}{2} g^{pk} g^{\alpha\beta} - g^{k\alpha} g^{p\beta}$ $(k, p, \alpha, \beta = 1, 2, ..., n)$ is a contravariant tensor of the fourth order, and $\left(\delta g_{\alpha\beta}\right)_{,p}$ $(\alpha, \beta, p = 1, 2, ..., n)$ is a contravariant tensor of the third order. The left part of the condition (49) represents the composition and the convolution of tensors.

Taking into account the value A^k (k=1,2,...,n) (30) it is easy to show that

$$\int_S \left[\delta\left(\sqrt{|g|}A^k\right) + \sqrt{|g|}\frac{\partial L^{\bullet}}{\partial\left(\dfrac{\partial g_{\alpha\beta}}{\partial p^k}\right)}\delta g_{\alpha\beta} \right] dS_k =$$

$$= \int_S \left\{ \sqrt{|g|}\frac{\partial L^{\bullet}}{\partial\left(\dfrac{\partial g_{\alpha\beta}}{\partial p^k}\right)} - \frac{\partial}{\partial p}\left[\sqrt{|g|}\left(\frac{1}{2}g^{kp}g^{\alpha\beta} - g^{k\alpha}g^{p\beta}\right)\right]\right\}\delta g_{\alpha\beta} dS_k - \tag{52}$$

$$-\int_S \sqrt{|g|}\left(\frac{1}{2}g^{kp}g^{\alpha\beta} - g^{k\alpha}g^{p\beta}\right)\frac{\partial\left(\delta g_{\alpha\beta}\right)}{\partial p^k} dS_k = 0 .$$

Let us multiply (49) by $\sqrt{|g|}\,dS_k$, sum by index k from one to n, and integrate the received equality by S. Then we shall get

$$\int_S \sqrt{|g|}\left(\frac{1}{2}g^{kp}g^{\alpha\beta} - g^{k\alpha}g^{p\beta}\right)\left(\delta g_{\alpha\beta}\right)_{,p} dS_k = 0 . \tag{53}$$

The sum of the equalities (52) and (53) gives

$$\int_S \left\{ \sqrt{|g|}\frac{\partial L^{\bullet}}{\partial\left(\dfrac{\partial g_{\alpha\beta}}{\partial p^k}\right)} - \frac{\partial}{\partial p^p}\left[\sqrt{|g|}\left(\frac{1}{2}g^{kp}g^{\alpha\beta} - g^{k\alpha}g^{p\beta}\right)\right] - \right.$$

$$\left. -\sqrt{|g|}\left[\left(\frac{1}{2}g^{kp}g^{l\beta} - g^{kl}g^{p\beta}\right)\Gamma^{\alpha}_{lp} + \left(\frac{1}{2}g^{kp}g^{\alpha l} - g^{k\alpha}g^{pl}\right)\Gamma^{\beta}_{lp}\right] \right\}\delta g_{\alpha\beta} dS_k = 0 . \tag{54}$$

That equality is fulfilled automatically if:

1) all $g_{\alpha\beta}$ $(\alpha,\beta = 1,2,...,n)$ are given functions on the surface S -

$$g_{\alpha\beta}\big|_S \; \textit{the given function} \; (\alpha,\beta = 1,2,...,n) \tag{55}$$

(at that $\delta g_{\alpha\beta}\big|_S = 0$ $(\alpha,\beta = 1,2,...,n)$ and (54) is true), or

2) $$\left\{ \frac{\partial L^{\bullet}}{\partial\left(\dfrac{\partial g_{\alpha\beta}}{\partial p^k}\right)} - \frac{1}{\sqrt{|g|}}\left[\sqrt{|g|}\left(\frac{1}{2}g^{kp}g^{\alpha\beta} - g^{k\alpha}g^{p\beta}\right)\right] - \right.$$

$$\left. -\left(\frac{1}{2}g^{kp}g^{l\beta} - g^{kl}g^{p\beta}\right)\Gamma^{\alpha}_{lp} - \left(\frac{1}{2}g^{kp}g^{\alpha l} - g^{k\alpha}g^{pl}\right)\Gamma^{\beta}_{lp} \right\} n_k = 0 , \tag{56}$$

where

$$n_k = \frac{dS_k}{dS} \quad (k = 1, 2, ..., n) \tag{57}$$

is a unit covariant normal vector to the hypersurface S, and

$$dS = \sqrt{g^{pq} dS_p dS_q} \; - \tag{58}$$

is an elementary area of the hypersurface S.

Using the boundary conditions (55) and (56) we can pose the following boundary problems:

1) To find a regular solution of the system of differential equations (38) relative to the unknown functions $g_{\alpha\beta}$ $(\alpha, \beta = 1, 2, ..., n)$, satisfying the boundary conditions of (55) in the domain V.

2) To find a regular solution of the system of differential equations of (38) relative to the unknown functions $g_{\alpha\beta}$ $(\alpha, \beta = 1, 2, ..., n)$, satisfying the boundary conditions of (56) in the domain V.

Using other limiting conditions relative to the arbitrary functions of (48) we can build other boundary conditions which satisfy the functions $g_{\alpha\beta}$ $(\alpha, \beta = 1, 2, ..., n)$ on the hypersurface, and pose corresponding boundary problems.

ინფორმატიკა

ხარისხის კრიტერიუმთა სივრცის მეტრიკული თვისებების გამოკვლევა მრავალკრიტერიული ოპტიმიზაციის ამოცანების ამოხსნის შემთხვევაში

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მათემატიკური თვალსაზრისით, ოპტიმალურ რეჟიმში ფუნქციონირებადი რეალური ობიექტის დაპროექტების პრობლემა დაიყვანება მრავალკრიტერიული ოპტიმიზაციის ამოცანის ამოხსნაზე, სადაც ხარისხის კრიტერიუმებად მისი ძირითადი მახასიათებლები (პარამეტრები) შეირჩევა. ამასთან, მრავალკრიტერიული ოპტიმიზაციის დღეისათვის არსებულ ყველა მეთოდში ხარისხის ვექტორული კრიტერიუმი, ძირითადად, განიხილება ევკლიდეს წრფივ სივრცეში. რეალურად, უმრავლეს შემთხვევაში, კრიტერიუმთა სივრცე არაევკლიდურია – იგი გამრუდებულია. ამიტომ მრავალკრიტერიული ოპტიმიზაციის ამოცანების გადაწყვეტა ევკლიდეს სივრცეში არ იძლევა შედეგებს, რომლებიც ადეკვატურად ასახავს რეალურ სისტემებში მიმდინარე პროცესებს.

იმისათვის, რომ დასაპროექტებელი ობიექტი ოპტიმალურ მოთხოვნებს რეალურად აკმაყოფილებდეს, ავტორების მიერ შემოთავაზებულია მრავალკრიტერიული ოპტიმიზაციის ამოცანების ამოხსნის სრულიად ახალი მიდგომა, რომელიც ეფუძნება ხარისხის კრიტერიუმთა სივრცის მეტრიკის განსაზღვრასა და ამ სივრცის ნებისმიერ ორ წერტილს შორის მანძილის შესაბამისი ინვარიანტის მოძებნას.

ხარისხის კრიტერიუმთა სივრცის მეტრიკული თვისებების გამოკვლევისა და მრავალკრიტერიული ოპტიმიზაციის ამოცანების გადაწყვეტის პროცესში მათი გამოყენების იდეა პირველად შემოთავაზებულ იქნა ნაშრომში [1], მაგრამ აღნიშნული იდეა, თავისი სირთულის გამო, დღემდე სრულად ვერ იქნა რეალიზებული. მსგავსი ამოცანების გადაწყვეტის დროს ხარისხის კრიტერიუმთა სივრცე ავტომატურად გაიგივებულია შესაბამისი მეტრიკის ევკლიდეს წრფივ სივრცესთან, რაც, თავის მხრივ, ვერ უზრუნველყოფს რეალურ სისტემებში მიმდინარე პროცესების ადეკვატურობას.

წინამდებარე ნაშრომში ხარისხის კრიტერიუმთა სივრცის მეტრიკული თვისებები პირველადაა გამოკვლეული, რისთვისაც გამოყენებულია ტენზორული ანალიზის, რიმანის გეომეტრიის, კერძოწარმოებულიანი დიფერენციალური განტოლებებისა და სხვათა მათემატიკური აპარატი. დასმულია სასაზღვრო ამოცანები მოვლენის მდგომარეობის n-განზომილებიანი სივრცის მეტრიკული ტენზორის კომპონენტების მიმართ, რომლებიც მისი თვისებების განსაზღვრის საშუალებას იძლევა. მეტრიკული ტენზორის ცოდნა უზრუნველყოფს მოვლენის მდგომარეობის ობიექტურ შეფასებასა და ოპტიმალური მდგომარეობის განსაზღვრას.

REFERENCES

1. *M.E. Salukvadze* (1979), Vector-Valued Optimization Problems in Control Theory. Academic Press, New York.
2. *L.P. Eisenhart* (1926), Riemannian Geometry. Princeton.
3. *R.Sh.Gogsadze* (1975), Ob odnoi gruppe negolonomnykh preobrazovanii i edinoe opisanie gravitatsionnogo i elektromagnitnogo polei. Tbilisi (in Russian).
4. *R.Sh.Gogsadze, V.K.Gogichaishvili* (2002), Matematikuri modelireba mikroelektronikashi [Mathematical modelling in microelectronics]. Tbilisi (in Georgian).
5. *R. Courant, D. Hilbert* (1952), Metody matematicheskoi fiziki, v. I, II. Moscow-Leningrad (in Russian).
6. *W. Pauli* (1958), Theory of Relativity. Pergamon Press, Oxford.

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