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Mathematics

Convergence Properties of φ-Summing Operators

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Dedicated to Professor Eusebio Corbacho Rosas on the occasion of his 60th birthday

ABSTRACT. By means of a sequence $\varphi_. = (\varphi_n)_{n\in N}$ of square-integrable functions, not necessarily orthonormal, on a positive measure space (Ω, ν) a notion of a $\varphi_.$-summing operator can be defined. We show that the class $\Pi_{\varphi_.}(X,Y)$ of all $\varphi_.$-summing operators acting between Banach spaces *X, Y* is non-trivial iff $\varphi_.$ is a 2-dominated sequence. It is shown also that if $\varphi_.$ is a 2-dominated sequence, then every 2-summing operator is $\varphi_.$*-summing. *© 2009 Bull. Georg. Natl. Acad. Sci.*

Key words: *2-summing operator, φ-summing operator, orthonormal sequence, 2-dominated sequence.*

Introduction

Let X, Y be normed spaces over the same field $\mathbf{K}$ of real or complex numbers and $L(X,Y)$ be the vector space of all continuous linear operators from *X* to *Y* endowed with the usual operator norm. For a normed space *X* with the dual space $X^* = L(X,\mathbf{K})$, we write $B_{X^*} = \{x^* \in X^* : \|x^*\| \le 1\}$.

For an operator $T \in L(X,Y)$, a number p, $0 < p < \infty$, and a natural number *n* we denote $\pi_p^{(n)}(T)$ the least constant $c \ge 0$ such that for each $(x_1,\dots,x_n) \in X^n$ the following inequality holds:

$$\left(\sum_{k=1}^{n}\|Tx_k\|^p\right)^{1/p} \le c \sup_{x^*\in B_{X^*}}\left(\sum_{k=1}^{n}|x^*(x_k)|^p\right)^{1/p}.$$

The operator *T* is called p-summing (or absolutely *p*-summing) if

$$\pi_p(T) := \sup_{n\ge 1}\pi_p^{(n)}(T) < \infty.$$

The set of all p-summing operators $T : X \to Y$ is denoted by $\Pi_p(X,Y)$. If $p \ge 1$, then the mapping $T \to \pi_p(T)$ is a norm on $\Pi_p(X,Y)$ and it is called p-summing norm.

This concept of a p-summing operator was introduced by A. Pietsch in 1966 and it played and continues to play an important role in Functional Analysis [1].

Let $\varphi_{\cdot} = (\varphi_n)_{n\in\mathbf{N}}$ be a sequence of square-integrable functions given on a positive measure space $(\Omega, \mathbf{A}, \nu)$ such that

$$\|\varphi_n\|_2 := \left(\int_\Omega |\varphi_n(\omega)|^2 \, d\nu(\omega)\right)^{1/2} > 0, \quad n = 1, 2, \ldots$$

For an operator $T \in L(X,Y)$ and a natural number *n* we denote by $\|T\|_{n,\varphi_{\cdot}}$ the least among the constants $c \geq 0$ such that for each $(x_1,\ldots,x_n) \in X^n$

$$\left(\int_\Omega \left\|\sum_{k=1}^n Tx_k\varphi_k(\omega)\right\|^2 d\nu(\omega)\right)^{1/2} \leq c \sup_{x^*\in B_{X^*}} \left(\sum_{k=1}^n |x^*(x_k)|^2\right)^{1/2}.$$

The operator *T* will be called $\varphi_{\cdot}$-summing (or $\varphi_{\cdot}$-bounding) [2] if

$$\|T\|_{\varphi_{\cdot}} := \sup_{n\geq 1}\|T\|_{n,\phi} < \infty.$$

The set of all $\varphi_{\cdot}$-summing operators $T : X \to Y$ will be denoted $\Pi_{\varphi_{\cdot}}(X,Y)$. The mapping $T \to \|T\|_{\varphi_{\cdot}}$ is a norm on $\Pi_{\varphi_{\cdot}}(X,Y)$ and it is called $\varphi_{\cdot}$-summing norm.

If $g_{\cdot} = (g_n)_{n\in\mathbf{N}}$ is a sequence of independent standard Gaussian random variables given on a probability space, then the class $\Pi_{\varphi_{\cdot}}(X,Y)$ of g-summing operators will coincide with the class of Gaussian-summing (or γ-summing) operators introduced in [3] (see also [4: (4.15.7)]).

If $r_{\cdot} = (r_n)_{n\in\mathbf{N}}$ is the sequence of Rademacher functions given on $[0,1]$ with the Lebesgue measure, then the class $\Pi_{r_{\cdot}}(X,Y)$ of $r_{\cdot}$-summing operators will coincide with the class $\Pi_{as}(X,Y)$ of almost summing operators introduced in [1].

If $\varphi_{\cdot} = (\varphi_n)_{n\in\mathbf{N}}$ is a sequence of independent identically distributed symmetric random variables given on a probability space, then the class $\Pi_{\varphi_{\cdot}}(X,Y)$ of $\varphi_{\cdot}$-summing operators appeared in [5].

It seems that the class of $\varphi_{\cdot}$-summing operators when $\varphi_{\cdot} = (\varphi_n)_{n\in\mathbf{N}}$ is an arbitrary orthonormal sequence was explicitly defined and studied in detail in [6, 7].

In [8] (where the notion of a $\varphi_{\cdot}$-summing operator presents implicitly) it was shown that if $\varphi_{\cdot}$ is an orthonormal basis in $L_2(\Omega,\nu;\mathbf{K})$, then for every pair X,Y of Banach spaces $\Pi_2(X,Y) = \Pi_{\varphi_{\cdot}}(X,Y)$ and $\|T\|_{\varphi_{\cdot}} = \pi_2(T),\ \forall T \in \Pi_2(X,Y)$. In [9] it is proved that if $\varphi_{\cdot}$ is a (not necessarily orthonormal) normalized unconditional basis in $L_2(\Omega,\nu;\mathbf{K})$, then for every pair X,Y of Banach spaces the equality $\Pi_2(X,Y) = \Pi_{\varphi_{\cdot}}(X,Y)$ remains true.

In [1, Theorem 12.12] it was shown that for any pair of Banach spaces X,Y

$$\Pi_{as}(X,Y) = \Pi_{g_{\cdot}}(X,Y) \quad \text{and} \quad \left(\frac{2}{\pi}\right)^{1/2} \|\cdot\|_{r_{\cdot}} \leq \|\cdot\|_{g_{\cdot}} \leq \|\cdot\|_{r_{\cdot}}.$$

It is known also that for any pair of Banach spaces X,Y and for any orthonormal sequence $\varphi_{\cdot} := (\varphi_n)_{n\in N}$ the inclusion $\Pi_{\varphi_{\cdot}}(X,Y) \subset \Pi_{g_{\cdot}}(X,Y)$ remains true and the inequality $\|T\|_{g_{\cdot}} \leq \|T\|_{\varphi_{\cdot}}$ holds (cf. [4: Theorem 4.15.3]; see also [10: Remark 3.10] where this fact is mentioned as known, but the proof is given too). In [6: 16] the considered result is also presented as known and is quoted in its connection [10: Remark 3.10] and [11: Theorem 5.5]).

In [2] it was obtained the following general result:

Theorem 1.1 *Let* X,Y *be normed spaces,* $\varphi_{\cdot} = (\varphi_k)_{k\in\mathbf{N}}$ *be a sequence of square integrable scalar functions given on a positive measure space* $(\Omega, \mathbf{A}, \nu)$ *such that*

$$\beta := \sup_n \frac{\sqrt{n}}{\left(\sum_{k=1}^{n} \|\varphi_k\|_2^2\right)^{1/2}} < \infty. \qquad (1)$$

Then:

(*a*) $\|T\|_{n,g.} \le \beta \|T\|_{n,\varphi.}$ $\forall n \in \mathbf{N}$, $\forall T \in L(X,Y)$.

(*b*) *We have* $\Pi_{\varphi.}(X,Y) \subset \Pi_{g.}(X,Y)$ and

$$\|T\|_{g.} \le \beta \|T\|_{\varphi.}, \ \forall T \in \Pi_{\varphi.}(X,Y).$$

As this was noted in [2], in general it may happen that for every given pair of Banach spaces (X,Y) and a sequence $(\varphi_k)_{k\in\mathbf{N}}$ satisfying (1) the class $\Pi_{\varphi.}(X,Y)$ is trivial, i.e. $\Pi_{\varphi.}(X,Y) = \{0\}$. In the present paper we show that the problem of non-triviality admits a complete solution in terms of 2-dominated sequences (see the next section for a definition). Namely, we have the following result:

Theorem 1.2 *Let* $\varphi. := (\varphi_n)_{n\in\mathbf{N}}$ *be a sequence of square integrable functions. Then:*

(*a*) *If for some pair* (X,Y) *of normed spaces the class* $\Pi_{\varphi.}(X,Y)$ *is not trivial, i.e.* $\Pi_{\varphi.}(X,Y) \ne \{0\}$, *then* $(\varphi_n)_{n\in\mathbf{N}}$ *is a 2-dominated sequence in* $L_2(\Omega,\nu;\mathbf{K})$.

(*b*) *If* $(\varphi_n)_{n\in\mathbf{N}}$ *is a 2-dominated sequence in* $L_2(\Omega,\nu;\mathbf{K})$, *then for every pair* X,Y *of normed spaces the inclusion* $\Pi_2(X,Y) \subset \Pi_{\varphi.}(X,Y)$ *holds and*

$$\|T\|_{\varphi.} \le h_{\varphi.}\pi_2(T), \ \forall T \in \Pi_2(X,Y),$$

where $h_{\varphi.}$ *is the 2-dominating constant of* $\varphi. = (\varphi_n)_{n\in\mathbf{N}}$.

In the following statements we give the descriptions of $\varphi.$-summing operators by means of boundedness and convergence properties.

Proposition 1.3. Let X,Y be normed spaces $\varphi. := (\varphi_n)_{n\in\mathbf{N}}$ a sequence of square integrable functions. Then for an operator $T: X \to Y$ the following statements are equivalent:

(*i*) $T \in \Pi_{\varphi.}(X,Y)$.

(*ii*) For every weakly 2-summable sequence $(x_n)_{n\in\mathbf{N}}$ in X the sequence $(Tx_n\varphi_n)_{n\in\mathbf{N}}$ is unconditionally bounded in $L_2(\Omega,\nu;Y)$.

(*iii*) For every weakly 2-summable sequence $(x_n)_{n\in\mathbf{N}}$ in X the sequence $\left(\sum_{k=1}^{n} Tx_k\varphi_k\right)_{n\in\mathbf{N}}$ is bounded in $L_2(\Omega,\nu;Y)$.

Theorem 1.4. *Let X be an arbitrary Banach space, Y a Banach space which does not contain a closed vector subspace isomorphic with* c_0 *and* $\varphi. := (\varphi_n)_{n\in\mathbf{N}}$ *a sequence of square integrable functions. Then for an operator* $T: X \to Y$ *the following statements are equivalent:*

(*i*) $T \in \Pi_{\varphi.}(X,Y)$.

(*ii*) *For every weakly 2-summable sequence* $(x_n)_{n\in\mathbf{N}}$ *in X the series* $\sum_n Tx_n\varphi_n$ *is unconditionally convergent in* $L_2(\Omega,\nu;Y)$.

(*iii*) *For every weakly 2-summable sequence* $(x_n)_{n\in\mathbf{N}}$ *in X the series* $\sum_n Tx_n\varphi_n$ *is convergent in* $L_2(\Omega,\nu;Y)$.

Finally, we note that the implication $(i) \Rightarrow (iii)$ of Theorem 1.4 is not true for $X = l_2$ and $Y = c_0$ when $\varphi. = (\varphi_n)_{n\in\mathbf{N}}$ is the Rademacher sequence [12].

2. Auxiliary definitions and proofs

For a positive measure space $(\Omega, \mathbf{A}, \nu)$ a number p, $0 < p < \infty$, and a normed space X we write $L_p(\Omega, \nu; X)$ for the space of all Borel measurable functions $\xi : \Omega \to X$ for which $\xi(\Omega)$ is a separable subset of X and

$$\|\xi\|_p := \left(\int_\Omega \|\xi(\omega)\|^p \, d\nu(\omega)\right)^{\frac{1}{p}} < \infty.$$

We denote by $L_{p,w}(\Omega, \nu; X)$ the set of all functions $\xi : \Omega \to X$ such that $x^* \circ \xi \in L_p(\Omega, \nu; \mathbf{K})$, $\forall x^* \in X^*$. It follows from closed graph theorem that

$$\|\xi\|_{p,w} := \sup_{x^* \in B_{X^*}} \|x^* \circ \xi\|_p < \infty, \ \forall \xi \in L_{p,w}(\Omega, \nu; X).$$

Clearly,

$$L_p(\Omega, \nu; X) \subset L_{p,w}(\Omega, \nu; X)$$

and

$$\|\xi\|_{p,w} \le \|\xi\|_p, \ \forall \xi \in L_p(\Omega, \nu; X).$$

If n is a natural number, $\Omega = \{1, \ldots, n\}$ and ν is the counting measure on Ω, then $L_p(\Omega, \nu; X) = L_{p,w}(\Omega, \nu; X) = X^n$ and in this case for each $(x_1, \ldots, x_n) \in X^n$:

$$\|(x_1, \ldots, x_n)\|_p = \left(\sum_{k=1}^{n} \|x_k\|^p\right)^{1/p},$$

$$\|(x_1, \ldots, x_n)\|_{p,w} = \sup_{x^* \in B_{X^*}} \left(\sum_{k=1}^{n} |x^*(x_k)|^p\right)^{1/p}$$

and

$$\|(x_1, \ldots, x_n)\|_{p,w} \le \|(x_1, \ldots, x_n)\|_p \le n^{\max(1,1/p)} \|(x_1, \ldots, x_n)\|_{p,w}.$$

If $\Omega = \mathbf{N}$ and ν is the counting measure on Ω, then $l_p(X) := L_p(\Omega, \nu; X)$ and $l_{p,w}(X) := L_{p,w}(\Omega, \nu; X)$.

In other words, $l_p(X)$ consists of all (infinite) sequences $x_\cdot = (x_n)_{n \in \mathbf{N}}$ of elements of X such that

$$\|x_\cdot\|_p = \left(\sum_{k=1}^{\infty} \|x_k\|^p\right)^{1/p} < \infty,$$

while an (infinite) sequence $x_\cdot = (x_n)_{n \in \mathbf{N}}$ of elements of X belongs to $l_{p,w}(X)$ iff it is weakly p-summable, i.e.

$$\sum_{n=1}^{\infty} |x^*(x_n)|^p < \infty, \ \forall x^* \in X^*.$$

We have also,

$$\|x_\cdot\|_{p,w} = \sup_{x^* \in B_{X^*}} \left(\sum_{n=1}^{\infty} |x^*(x_n)|^p\right)^{1/p} < \infty \ \ \forall x_\cdot = (x_n)_{n \in \mathbf{N}} \in l_{p,w}(X).$$

Remark 2.1. Let X, Y be normed spaces and $0 < p < \infty$.

(*a*) *It is an important consequence of Dvoretzky's theorem that the equality* $l_p(X) = l_{p,w}(X)$ *holds iff X is finite-dimensional* [13: 104].

(*b*) *For an operator* $T \in L(X,Y)$ *we have* $T \in \Pi_p(X,Y)$ *iff* $(Tx_n) \in l_p(Y)$ *for every* $(x_n) \in l_{p,w}(X)$ (Pietsch).

If $1 \le p < \infty$, then it is possible to give an internal description of weakly p-summable sequences. We need such a description when p=2 and p=1.

For a sequence $\varphi_. = (\varphi_n)_{n\in\mathbf{N}}$ in a normed space E and a natural number n let $h_{n,\varphi_.}$ be the least among the constants $h \geq 0$ such that

$$\left\|\sum_{k=1}^{n} t_k \varphi_k\right\| \leq h\left(\sum_{k=1}^{n} |t_k|^2\right)^{\frac{1}{2}}, \ \forall (t_1,\cdots,t_n) \in \mathbf{K}^n.$$

We say that the sequence $\varphi_.$ is 2-dominated if $h_{\varphi_.} := \sup_{n\in\mathbf{N}} h_{n,\varphi_.} < \infty$ and $h_{\varphi_.}$ we call the 2-dominating constant of $\varphi_.$.

Remark 2.2. *It is known that a sequence $\varphi_.$ in a normed space E is a 2-dominated sequence iff $\varphi_.$ is a weakly 2-summable sequence. Moreover, for each 2-dominated $\varphi_.$ we have: $h_{\varphi_.} = \|\varphi_.\|_{2,w}$.*

A sequence $\varphi_.$ in a Hilbert space $\left(E,(\cdot\,|\,\cdot)\right)$ is a 2-dominated sequence iff for some finite positive constant c

$$\sum_{n=1}^{\infty} |(\varphi_n \,|\, \psi)|^2 \leq c^2 \|\psi\|^2, \ \forall \psi \in E.$$

In [14: Definition 3.1.2] a 2-dominated sequence in a Hilbert space is called a Bessel sequence.

Observe that if $\varphi_. = (\varphi_n)_{n\in\mathbf{N}}$ is an orthonormal sequence in a Hilbert space E, then it is 2-dominated sequence and $h_{\varphi_.} = 1$.

For a sequence $\varphi_. = (\varphi_n)_{n\in\mathbf{N}}$ in a normed space E and a natural number n let $b_{n,\varphi_.}$ be the least among the constants $b \geq 0$ such that

$$\left\|\sum_{k\in\Delta} \varphi_k\right\| \leq b, \ \forall \Delta \subset \{1,\cdots,n\}.$$

We say that the sequence $\varphi_.$ is unconditionally bounded if

$$b_{\varphi_.} := \sup_{n\in N} b_{n,\varphi_.} < \infty$$

and $b_{\varphi_.}$ we call the uc-bounding constant of $\varphi_.$.

Remark 2.3. *Let $\varphi_. = (\varphi_n)_{n\in\mathbf{N}}$ be a sequence in a Banach space E.*

(*a*) *It is known that $\varphi_.$ is unconditionally bounded iff is a weakly 1-summable sequence. Moreover, for each unconditionally bounded $\varphi_.$ we have: $b_{\varphi_.} = \|\varphi_.\|_{1,w}$.*

(*b*) *If the series $\sum_n \varphi_n$ is unconditionally convergent, then the sequence $\varphi_.$ is unconditionally bounded.*

(*c*) *(Bessaga-Pelczynski's theorem [14: Theorem 2.3.3]) For every unconditionally bounded sequence $\varphi_.$ the series $\sum_n \varphi_n$ is unconditionally convergent iff E does not contain a closed vector subspace isomorphic to c_0.*

Lemma 2.4. *Let $(\varphi_n)_{n\in\mathbf{N}}$ be a 2-dominated sequence in $L_2(\Omega,\nu;\mathbf{K})$, H be a Hilbert space.*

(*a*) $\forall n \in \mathbf{N}$ *and* $\forall z_1,\cdots,z_n \in H$,

$$\left\|\sum_{k=1}^{n} z_k \varphi_k\right\|_2 \leq h_{\varphi_.}\left(\sum_{k=1}^{n} \|z_k\|^2\right)^{\frac{1}{2}};$$

(*b*) *If $(z_n) \in l_2(H)$, then the series $\sum_{k=1}^{\infty} z_k \varphi_k$ is unconditionally convergent in $L_2(\Omega,\nu;H)$ and the inequality*

$$\left\|\sum_{k=1}^{\infty} z_k \varphi_k\right\|_2 \leq h_{\varphi_.}\left(\sum_{k=1}^{\infty} \|z_k\|^2\right)^{\frac{1}{2}}$$

holds.

Proof. (*a*) is easy to see.

(*b*) Fix a bijection $\pi:\mathbf{N}\to\mathbf{N}$ and natural numbers $n<m$. Since $(\varphi_{\pi(n)})$ is also a 2-dominated sequence with the same constant c, we have:

$$\left\|\sum_{k=n}^{m} z_{\pi(k)}\varphi_{\pi(k)}\right\|_2 \le h_{\varphi_\cdot}\left(\sum_{k=n}^{m}\left\|z_{\pi(k)}\right\|^2\right)^{\frac{1}{2}}$$

consequently,

$$\lim_{n,m\to\infty}\left\|\sum_{k=n}^{m} z_{\pi(k)}\varphi_{\pi(k)}\right\|_2 = 0.$$

Then, since $L_2(\Omega,\nu;H)$ is complete, we have that $\sum_{k=1}^{\infty} z_k\varphi_k$ is unconditionally convergent and the needed inequality holds as well.

Corollary 2.5. *Let* (φ_n) *be a 2-dominated sequence in* $L_2(\Omega,\nu;\mathbf{K})$ *, X normed space, H a Hilbert space.*

(*a*) *If* $T\in\Pi_2(X,H)$ *and* $(x_n)\in l_{2,w}(X)$, *then the series* $\sum_{k=1}^{\infty} Tx_k\varphi_k$ *is unconditionally convergent in* $L_2(\Omega,H)$ *and the following inequality is true*

$$\left\|\sum_{k=1}^{\infty} Tx_k\varphi_k\right\|_2 \le h_{\varphi_\cdot}\pi_2(T)\left\|(x_k)\right\|_{2,w}.$$

(b) $\Pi_2(X,H)\subset\Pi_{\varphi_\cdot}(X,H)$ *and* $\|T\|_{\varphi_\cdot}\le h_{\varphi_\cdot}\pi_2(T),\ \forall T\in\Pi_2(X,H).$

Proof. (*a*) Let $T\in\Pi_2(X,H)$ and $(x_n)\in l_{2,w}(X)$. Then $(Tx_n)\in l_2(H)$ and $\|(Tx_n)\|_2\le\pi_2(T)\|(x_n)\|_{2,w}$. From Lemma 2.4 (*b*) we have that the series $\sum_{k=1}^{\infty} Tx_k\varphi_k$ is unconditionally convergent in $L_2(\Omega,H)$ and

$$\left\|\sum_{k=1}^{\infty} Tx_k\varphi_k\right\|_2 \le h_{\varphi_\cdot}\left(\sum_{k=1}^{\infty}\left\|Tx_k\right\|^2\right)^{\frac{1}{2}} \le h_{\varphi_\cdot}\pi_2(T)\left\|x_\cdot\right\|_{2,w}.$$

(*b*) follows from (*a*)

Proof of Theorem 1.2.

(*a*) Fix $T\in\Pi_{\varphi_\cdot}(X,Y)$, $\|T\|=1$, a vector $x\in S_X$, $Tx\ne 0$, a natural number n and a finite sequence $t_1,\ldots,t_n$ of scalars. Since $T\in\Pi_{\varphi_\cdot}(X,Y)$, we can write

$$\|Tx\|\left\|\sum_{k=1}^{n} t_k\varphi_k\right\|_2 = \left\|\sum_{k=1}^{n} T(t_kx)\varphi_k\right\|_2 \le \|T\|_{\varphi_\cdot}\left\|(t_1x,\ldots,t_nx)\right\|_{2,w} = \|T\|_{\varphi_\cdot}\left(\sum_{k=1}^{n}|t_k|^2\right)^{\frac{1}{2}}.$$

From this inequality, since $x\in S_X$ with $Tx\ne 0$ is arbitrary and $\|T\|=1$, we get:

$$\left\|\sum_{k=1}^{n} t_k\varphi_k\right\|_2 \le \|T\|_{\varphi_\cdot}\left(\sum_{k=1}^{n}|t_k|^2\right)^{\frac{1}{2}}.$$

Hence, $\varphi_\cdot=(\varphi_n)_{n\in\mathbf{N}}$ is a 2-dominated sequence with $h_{\varphi_\cdot}\le\|T\|_{\varphi_\cdot}$.

(*b*) Fix $T\in\Pi_2(X,Y)$, a natural number n, a finite sequence $(x_1,\ldots,x_n)\in X^n$ and write $\xi=\sum_{k=1}^{n} x_k\varphi_k$. Since $\varphi_\cdot=(\varphi_n)_{n\in\mathbf{N}}$ is a 2-dominated sequence, we have:

$$\left\|x^*\circ\xi\right\|_2 = \left\|\sum_{k=1}^{n} x^*(x_k)\varphi_k\right\|_2 \le h_{\varphi_\cdot}\left(\sum_{k=1}^{n}|x^*(x_k)|^2\right)^{\frac{1}{2}},\ \forall x^*\in X^*$$

and so,

$$\|x^* \circ \xi\|_2 \le h_{\varphi.} \|(x_1,\dots,x_n)\|_{2,w}, \ \forall x^* \in B_{X^*}. \tag{2}$$

From $T \in \Pi_2(X,Y)$ it is standard to get:

$$\left\|\sum_{k=1}^{n} Tx_k \varphi_k\right\|_2 = \|T \circ \xi\|_2 \le \pi_2(T) \sup_{x^* \in B_{X^*}} \|x^* \circ \xi\|_2 .$$

From the last relation and (2) we obtain:

$$\left\|\sum_{k=1}^{n} Tx_k \varphi_k\right\|_2 \le \pi_2(T) h_{\varphi.} \|(x_1,\dots,x_n)\|_{2,w}.$$

Hence, $T \in \Pi_{\varphi.}(X,Y)$ and $\|T\|_{\varphi.} \le \pi_2(T) h_{\varphi.}$. □

The implication $(i) \Rightarrow (ii)$ of the following assertion contains an improvement of Theorem 1.2(b).

Theorem 2.6. *For a given sequence* $\varphi. = (\varphi_n)_{n\in\mathbf{N}}$ *the following conditions are equivalent:*

(i) $(\varphi_n)_{n\in\mathbf{N}}$ *is a 2-dominated sequence in* $L_2(\Omega,\nu;\mathbf{K})$.

(ii) *If X is a normed space, Y is a Banach space, then there exists a finite constant* $c \ge 0$ *such that for each* $T \in \Pi_2(X,Y)$, *and for every weakly 2-summable sequence* $(x_n)_{n\in\mathbf{N}}$ *in X the series* $\sum_k Tx_k\varphi_k$ *is unconditionally convergent in* $L_2(\Omega,\nu;Y)$ *and*

$$\left\|\sum_{k=1}^{\infty} Tx_k \varphi_k\right\|_2 \le c\left(\sum_{k=1}^{\infty}\|Tx_k\|^2\right)^{\frac{1}{2}} \le c\pi_2(T)\|(x_k)\|_{2,w}, \ \forall (x_n) \in l_{2,w}(X).$$

(iii) *For every pair* X,Y *of normed spaces we have* $\Pi_2(X,Y) \subset \Pi_{\varphi.}(X,Y)$ *and there exists a finite constant* $c \ge 0$ *such that*

$$\|T\|_{\varphi.} \le c\pi_2(T), \ \forall T \in \Pi_2(X,Y).$$

(iv) *For every pair* X,Y *of normed spaces we have the inclusion* $\Pi_2(X,Y) \subset \Pi_{\varphi.}(X,Y)$.

Proof. $(i) \Rightarrow (ii)$. Take $T \in \Pi_2(X,Y)$ then, by Pietsch theorem, *T* admits a factorization $T = T_2T_1$ such that $T_1 \in \Pi_2(X,H)$, $T_2 \in L(H,Y)$, $\pi_2(T_1) \le \pi_2(T)$ and $\|T_2\| \le 1$.

From Corollary 2.5 we have that (ii) is true for T_1. This means that if $(x_n) \in l_{2,w}(X)$ then $\sum_k T_1x_k\varphi_k$ is unconditionally convergent in $L_2(\Omega,\nu;H)$ and

$$\left\|\sum_{k=1}^{\infty} T_1x_k \varphi_k\right\|_2 \le c\pi_2(T_1)\|x.\|_{2,w},$$

where *c* is the 2-dominating constant of $\varphi. = (\varphi_n)_{n\in\mathbf{N}}$. As $T_2 \in L(H,Y)$ the series $\sum_k T_2T_1x_k\varphi_k$ is unconditionally convergent in $L_2(\Omega,\nu;Y)$ and we have the inequality

$$\left\|\sum_{k=1}^{\infty} T_2T_1x_k \varphi_k\right\|_2 \le \|T_2\|\cdot\left\|\sum_{k=1}^{\infty} T_1x_k \varphi_k\right\|_2 \le c\pi_2(T_1)\cdot\|x.\|_{2,w} \le c\pi_2(T)\cdot\|x.\|_{2,w}.$$

Therefore,

$$\left\|\sum_{k=1}^{\infty} Tx_k \varphi_k\right\|_2 \le c\pi_2(T)\cdot\|x.\|_{2,w}.$$

(iii) follows from (ii).

$(iii) \Rightarrow (iv)$ is evident.

$(iv) \Rightarrow (i)$. Take $X = Y = \mathbf{K}$ and let $T: \mathbf{K} \to \mathbf{K}$ be the identity mapping. Then $T \in \Pi_2(\mathbf{K},\mathbf{K})$. Hence $T \in \Pi_{\varphi.}(\mathbf{K},\mathbf{K})$ and this clearly implies that (φ_n) is a 2-dominated sequence in $L_2(\Omega,\nu;\mathbf{K})$.

Proof of Proposition 1.3.

$(i) \Rightarrow (ii)$.

Fix $(x_n) \in l_{2,w}(X)$ and a finite non-empty $A \subset \mathbf{N}$. Then as $T \in \Pi_{\varphi_\cdot}(X,Y)$ we have

$$\left\|\sum_{k\in A} Tx_k\varphi_k\right\|_2 = \left\|\sum_{k=1}^{m} Tx_k'\varphi_k\right\|_2 \le \|T\|_{\varphi_\cdot} \cdot \sup_{\|x^*\|\le 1}\left(\sum_{k=1}^{m}|x^*(x_k')|^2\right)^{\frac{1}{2}} = \|T\|_{\varphi_\cdot} \cdot \sup_{\|x^*\|\le 1}\left(\sum_{k\in A}|x^*(x_k)|^2\right)^{\frac{1}{2}} \le \|T\|_{\varphi_\cdot} \cdot \|x_\cdot\|_{2,w},$$

where $m = \max A$ and $x_k' = x_k$ if $k \in A$ zero in other cases.

Hence, the sequence $(Tx_k\varphi_k)$ is unconditionally bounded in $L_2(\Omega,\nu;Y)$ (with a uc-bounding constant $\le \|T\|_{\varphi_\cdot} \cdot \|x_\cdot\|_{2,w}$).

$(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$.

Let us introduce a normed space in the following way.

$$B_{\varphi_\cdot}(Y) = \{(y_n) \in Y^{\mathrm{N}} \mid \; |||(y_n)||| := \sup_n \left\|\sum_{k=1}^{n} y_k\varphi_k\right\|_2 < \infty\}.$$

It can be seen that if Y is a Banach space, then $(B_{\varphi_\cdot}(Y), |||\cdot|||)$ is a Banach space too.

We can suppose without loss of generality that Y is a Banach space. Let us define the operator

$$\tilde{T} : l_{2,w}(X) \to B_{\varphi_\cdot}(Y)$$

which sends (x_n) to (Tx_n). It is easy to see that $\tilde{T}$ is linear and has closed graph. By closed graph theorem $\tilde{T}$ is continuous, i.e.

$$|||\tilde{T}x_\cdot||| \le \|\tilde{T}\| \cdot \|x_\cdot\|_{2,w}, \;\; \forall x_\cdot \in l_{2,w}(X).$$

The last inequality implies that T is $\varphi_\cdot$-summing and $\|T\|_{\varphi_\cdot} \le \|\tilde{T}\|$. □

Proof of Theorem 1.4.

$(i) \Rightarrow (ii)$.

Fix $(x_n) \in l_{2,w}(X)$; from Proposition 1.3 we have that the sequence $(Tx_n\varphi_n)_{n\in\mathbf{N}}$ is unconditionally bounded in $L_2(\Omega,\nu;Y)$, since Y does not contain copies of c_0, by Hoffmann-Jorgensen-Kwapien's theorem (see [13: Theorem 5.6.1]) $L_2(\Omega,\nu;Y)$ also does not contain copies of c_0; then by Bessaga-Pelczynski theorem the series $\sum_n Tx_n\varphi_n$ converges unconditionally in $L_2(\Omega,\nu;Y)$.

$(ii) \Rightarrow (iii)$. This is obvious.

$(iii) \Rightarrow (i)$. This follows from implication $(iii) \Rightarrow (i)$ of Proposition 1.3. □

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მათემატიკა

კრებადობის თვისებები φ-შემკრები ოპერატორებისათვის

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ეძღვნება პროფესორ ეისებიო კორბაჩოს დაბადების 60 წლისთავს

კვადრატულად ინტეგრებადი ფუნქციების ზოგადი $\varphi = (\varphi_n)_{n\in \mathbf{N}}$ მიმდევრობისათვის შემოტანილია φ-შემკრები ოპერატორის ცნება. ნაჩვენებია, რომ ბანახის X და Y სივრცეებს შორის მოქმედი φ-შემკრები ოპერატორების $\Pi_{\varphi.}(X,Y)$ კლასი არატრივიალურია მაშინ და მხოლოდ მაშინ, როდესაც φ მიმდევრობა დომინირებადია. ნაჩვენებია, აგრეთვე, რომ თუ φ დომინირებადია, მაშინ ნებისმიერი 2-შემკრები ოპერატორი φ-შემკრებიცაა.

REFERENCES

1. *J. Diestel, H. Jarchow, A. Tong* (1995), Absolutely summing operators. Cambridge University Press, Cambridge
2. *V. Tarieladze, R. Vidal* (2003), Acta Mathematica Vietnamica. **28**: 225-240.
3. *W. Linde, A. Pietsch* (1974), Theor. Probability Appl. **19**: 445-460.
4. *A. Pietsch, J. Wenzel* (1998), Orthonormal systems and Banach space geometry. Cambridge University Press, Cambridge, 553 p.
5. *O. Blasco, V. Tarieladze, R. Vidal* (2000), Georgian Math. J., 7**:** 245-268.
6. *F. Baur* (1997), Banach operator ideals generated by orthonormal systems. Inaugural -Dissertation, Universität Zurich, 72 p.
7. *F. Baur* (1998), Arch. Math. **71**: 465-471.
8. *M. Defant, M. Junge* (1992), Math. Nachr. **158**: 233-240.
9. *E. Corbacho, V. Tarieladze, R. Vidal* (2008), Riesz bases and *-summing operators. Preprint.
10. *S. Geis, M. Junge* (1995), J. Approximation Theory **82**: 399-433.
11. *J. Seigner* (1995), Über eine Klasse von Idealnormen, die mit Orthonormalsystemen gebildet sind., Dissertation, Friedrich-Schiller-Universität, Jena.
12. *Nguyen Duy Tien, V. Tarieladze, R. Vidal* (1998), Bull. Polish Acad. Sci. **46**, 365-375.
13. *N.N. Vakhania, V.I. Tarieladze, S.A. Chobanyan* (1987), Probability distributions on Banach Spaces, D. Reidel Publishing Company, Dordrecht, 482 p.
14. *O. Christensen* (2008), Frames and bases. An introductory course. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, MA. 313 p.

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