ᲡᲐᲥᲐᲠᲗᲕᲔᲚᲝᲡ ᲛᲔᲪᲜᲘᲔᲠᲔᲑᲐᲗᲐ ᲔᲠᲝᲕᲜᲣᲚᲘ ᲐᲙᲐᲓᲔᲛᲘᲘᲡ ᲛᲝᲐᲛᲑᲔ, ტ. 3, №3, 2009
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 3, no. 3, 2009

Mathematics

On a Probability Problem of Lewis Carroll

Nicholas Vakhania

Academy Member, Niko Muskhelishvili Institute of Computational Mathematics, Tbilisi

ABSTRACT. We give the solution of the general version of Lewis Carroll's probability problem No. 72 for any number of black or white counters initially in the bag, and any number of non-random black or white counters that could be put into the bag in addition. The result for the general version is given as formula (4). *© 2009 Bull. Georg. Natl. Acad. Sci.*

Key words: *black and white counters, additional (non-random) counters, total probability formula, general version of Lewis Carroll's problem.*

In memory of the person who was a widely
known writer Lewis Carroll and a good
mathematician Charles Lutwidge Dodgson.

Lewis Carroll (1832-1898) is the pen-name of an English mathematician, professor of Oxford University, and writer C.L. Dodgson, whose literary legacy is well-known. Among his literary works most widely known is "Alice in Wonderland" which has been published many times and translated into many languages (including more than one translation in Georgian).

In the first part of this note we speak on one of the 13 probability problems, namely the problem No.72, from Lewis Carroll's "Pillow Problems". We will speak on the proof, and give some comments on the approach to this problem by Lewis Carroll himself (together with some of our comments). In the second part we give a generalized version of this problem for the case of any number $n > 1$ of black or white counters initially in the bag and any number $b \geq 0$ black and $w \geq 0$ white counters that might be put into the bag in addition (in the case of Lewis Carroll's original problem $n = 2$, $b = 1$, $w = 0$).

As it is well-known, the Bayes' formula, as well as the total probability formula, has many applications. These theorems are in fact easy statements, and their applications should not be connected with any misunderstandings. But still, apparently just because of their many applications, these formulas often were used incorrectly. Strange examples are the attempts of application of Bayes' formula to reliability of witness' testimonies in the old judicial practice of some countries. Neither the formula of the total probability was free of illegitimate applications.

The problem to be considered in the first part of this note is the widely disputable example of the application of the total probability formula to the above mentioned problem No.72 of Lewis Carroll. We give this problem in the original formulation of Lewis Carroll, and give Lewis Carroll's original proof as well.

This problem, as it is formulated by Lewis Carroll, does not have solution (the conditions are not enough to obtain solution). The author of this note inclines to think that the formulation and the proof of this problem could be understood as a kind of a humorous reaction of Lewis Carroll to the cases of illegitimate application of Bayes' and the total probability formulas. This seems to be hinted also by Lewis Carroll himself choosing an involved proof instead of a quite simple one. However, this involved proof turned out to be of more interest than the standard one which is given with our comment after the formulation of the problem.

Formulation of Lewis Carroll's problem. A bag contains 2 counters, as to which nothing is known except that each is either black or white. Ascertain their colors without taking them out of the bag.

Clearly, it is impossible to have the deterministic solution (it is explicitly given in the formulation of the problem that each counter of the two can be either black or white). Therefore, the probabilistic setting is to be meant, and thus the probabilities for all the three possible combinations of colors *BB, BW, WW* of the two counters are to be found. We note that here and everywhere below *BB* and *WW* denotes two black and, respectively, two white counters and *BW* denotes the union of black and white and white and black counters.

Therefore now it should be quite clear that *P*(both counters are black) = *P*(both counters are white) $=1/4$ and *P*(one counter is black, the other is white)$=1/4+1/4=1/2$. In addition, the standard use of the total probability formula, taking as hypotheses *BB, BW, WW* easily gives that the probability of drawing a black counter from the common bag is equal to $1/2$. Indeed, $P(B)=(2/2)\cdot(1/4)+(1/2)\cdot(1/2)=1/2$ (this result follows also from the symmetry of the conditions with respect to black and white counters).

Thus, the problem is solved quite easily.

However, Lewis Carroll reasons otherwise. His original proof is given below (in fact, without any alteration).

The original proof. We know that, if a bag contained 3 counters, 2 being black and one white, the chance of drawing a black one would be 2/3; and that any *other* state of things would *not* give this chance.

Now the chances that the given bag contains (α) *BB*, (β) *BW*, (γ) *WW*, are respectively 1/4, 1/2, 1/4.

Add a black counter.

Then the chances, that it contains (α) *BBB,* (β) *BWB,* (γ) *BWW*, are, as before, 1/4, 1/2, 1/4.

Hence the chance of now drawing a black one,

$$=1\cdot\frac{1}{4}+\frac{2}{3}\cdot\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{4}=\frac{2}{3}. \qquad (*)$$

Hence the bag now contains *BBW* (since any *other* state of things would *not* give this chance).

Hence, before the black counter was added, the bag contained BW, i.e. one black counter and one white. Q.E.F. THE END.

Again, Lewis Carroll goes on, using non-probabilistic terms in the final conclusion while the solution was given in probabilistic terms: "Hence, before the black counter was added, the bag contained BW", instead of saying: "Hence, before the black counter was added, the probability of the composition BW of the counters was 1/2".

Lewis Carroll's proof of his problem No.72 is really ambiguous, and it has been tacitly accepted that the proof was incorrect. However, it seems that no clear and convincing proof of the incorrectness has been given, and the question "where is the fallacy" continues to be of interest and even challenging for many years already.

Remark. The author of this note thinks that essentially there is no fallacy in the Lewis Carroll's proof. We could say it more definitely if we had not felt the pressure of the opposite opinion on this matter in all published papers concerning this problem.

Our aim now in this note is to give a generalized version of the Lewis Carroll's problem.

Before passing to this item, we want to note that the way Lewis Carroll solved his problem is not a general method, it is not justified for all values of counters initially in the bag. If, for example, we have three counters in the bag, instead of two, Lewis Carroll's method will not give the result. The reason will be clear when we start to discuss this problem, and we start with the following preparatory remarks.

Let us have $n>1$ bags each containing one black and one white counter, identical except their colors. We draw randomly one counter from each of these n bags, and put them into a new common bag. It is easy to see that: 1) the probability of drawing of black or white counters from any of the n bags is the same number 1/2; 2) the common bag will contain n counters, and the probability of drawing the black or white counter from the common bag is the same number 1/2. Therefore, all 2^n possibilities of combinations of black and white colors for counters will have the same probability $1/2^n$ (and thus there will be no need to use the vague "principle of insufficient reason"); 3) the black counters among all the n can be of any number k, $k = 0,1,...,n$; 4) k black counters among all n can be chosen by C_n^k different ways, and the probability of any such choice is k/n, $k = 0,1,...,n$.

After these preparatory remarks, now we use the total probability formula in its standard form. We choose the hypotheses $H_0, H_1, \ldots, H_n$, where H_i ($i = 0,1,...,n$) means that there are i white and, consequently, $n-i$ black counters in the bag. It is easily seen that the total probability formula with these hypotheses gives the following expression for the probability of drawing a black counter from the common bag:

$$P(B) = \frac{1}{n2^n}[n + (n-1)C_n^1 + (n-2)C_n^2 + \cdots + 2C_n^{n-2} + C_n^{n-1}]. \tag{1}$$

It is also easy to see that after putting b black and w white counters into the bag, we get the following expression for the equality (1):

$$P(B) = \frac{1}{2^n(n+b+w)}[n + b + (n+b-1)C_n^1 + (n+b-2)C_n^2 + \cdots + (b+1)C_n^{n-1} + bC_n^n. \tag{2}$$

We note in passing that equality (2) for $n = 2$, $b = 1$, $w = 0$ coincides with equality (*) in the Lewis Carroll's proof.

Taking into account that the last two summands in the right-hand side of equality (2) can be written as $[n + b - (n-1)]C_n^{n-1} + (n + b - n)C_n^n$ (and the previous summands, correspondingly), we can write the right hand side of equality (2) as the following difference of the two fractions:

$$P(B) = \frac{(n+b)[1 + C_n^1 + C_n^2 + \cdots + C_n^n]}{2^n(n+b+w)} - \frac{C_n^1 + 2C_n^2 + \cdots + (n-1)C_n^{n-1} + nC_n^n}{2^n(n+b+w)}. \tag{3}$$

Then we calculate the two fractions in this equality separately. Using the binomial formula for the first fraction and the elementary equality $kC_n^k = nC_{n-1}^{k-1}$ for the second fraction, we get from (3) the following final expression for $P(B)$:

$$P(B) = \frac{n+2b}{2(n+b+w)}. \tag{4}$$

If $n = 2$, $b = 1$ and $w = 0$ formula (4) gives the Lewis Carroll's case $P(B) = 2/3$. If $b = w = 0$ then $P(B) = 1/2$ as it should be. We note also that for any pair of natural numbers b and w, $P(B) = 1/2$ for all $n \geq 2$ if and only if $b = w$. It is interesting also to note that formula (4) includes also the cases of $n = 0$ and $n = 1$ with any $b \geq 0$ and $w \geq 0$. If we take $b = n$, $w = 0$, then $P(B) = 3/4$ for any $n > 0$.

Acknowledgements. The author expresses his gratitude to Professor E. Seneta for the help in providing with some original materials including, at our request, two of his own papers on the subject around this note. We express our gratitude also to our local collegue G. Chelidze for useful discussions with him on the matter of this note.

მათემატიკა

ლუის კეროლის ერთი ალბათური ამოცანის შესახებ

ნ. ვახანია

აკადემიკოსი, ნიკო მუსხელიშვილის გამოთვლითი მათემატიკის ინსტიტუტი, თბილისი

მოცემულია ლუის კეროლის ამოცანის ზოგადი ვარიანტის ამოხსნა ყუთში თავიდან მოთავსებულ თეთრ ან შავ ბურთულათა ნებისმიერი რაოდენობისა და შემდეგ ჩამატებული ნებისმიერი რაოდენობების თეთრი და შავი ბურთულებისათვის. ზოგადი ვარიანტის პასუხი მიღებულია ფორმულა (4)-ის სახით.

REFERENCES

1. *F. Abeles*, ed. (1993), The Mathematical Pamphlets of Charles L.Dodgson. Univ. Virginia Press, Charlottsville.
2. *T. Beale* (1973), C.L. Dodgson: Mathematician. In: Mr Dodgson: Nine Lewis Carroll. Studies with a Companion-Guide to the Alice at Longleat Exhibition, 26-33. The Lewis Carroll Society.
3. *G. Boole* (1854), The laws of thought. Makmillan, London. (Reprinted by Dover. New York. 1958. Chapter 16 is entitled “On the Theory of Probabilities”, and chapter 17 and 18 deal with a “General method for the solution…” of probability problems).
4. *L. Carroll* (1958), Pillow Problems and a Tangled Tale. Dover. New York (The first part is a reprinting of the 4th edition of Pillow Problems, which has a preface written by Dodgson in March, 1895. The introduction to the 1st edn of Curiosa Mathematica, part II: Pillow Problems is dated May, 1893. The 1958 book is volume 2 of the Mathematical Recreations of Lewis Carroll published by Dover, in which volume 1 consists of symbolic Logic and The Game of Logic).
5. *L. Carroll* (1982), The Penguin Complete Lewis Carroll. Penguin. Harmondsworth (This edition first published as *The complete works of Lewis Carroll*, Nonesuch Press, London, 1939).
6. *A. Clark* (1979), Lewis Carroll: A Biography. Schocken Books, New York.
7. *M.W. Crofton* (1885), Probability. Encyclopedia Britannica (9th ed.), **19:** 768-788.
8. *D.B. Eperson* (1933), Lewis Carroll, mathematician. Math. Gazette, 1933, **17**: 92-100.
9. *J. Gabor* (1986), Paradoxes in Probability Theory and Mathematical Sciences. Dordrecht: Reidel.
10. *P. Heath* (1967), Carroll, Lewis. Encyclopedia of Philosophy. Crowell-Collier-Macmillan: New York.
11. *D. Hudson* (1954), Lewis Carroll. Constable, London (there is a revised 1977 edition).
12. *A.N. Kolmogorov and A.P. Yushkevich*, Eds.(1978), Mathematics in 19th Century, Moscow.
13. *E. Seneta* (1984), Lewis Carroll as a probabilist and mathematician. Math. Scientist. 79-94.
14. *E. Seneta* (1993), Lewis Carroll’s “Pillow Problems” on the 1993 Centenary. Statistical Science, **8**, No.2: 180-186.
15. *J. Stern*,ed. (1981), Lewis Carroll’s library. Carroll studies No.5. The Lewis Carroll Society of North America, Univ. Virginia Press, Charlottesville.
16. *W. Weaver* (1954), The mathematical manuskripts of Lewis Carroll. Proceedings of the American Philosofical Society, **98:** 371-381.
17. *W. Weaver* (1956), Lewis Carroll: Mathematician. Scientific American, **194:** 116-128.

Received July, 2009