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Mathematics

Maltsev Bases and Triangular Representations of Tensor Products of Abelian Groups

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ABSTRACT. The paper is dedicated to the construction of Maltsev bases and triangular representations of tensor products of Abelian groups. *© 2010 Bull. Georg. Natl. Acad. Sci.*

Key words: *Maltsev base, triangular representation, tensor product.*

Introduction

The calculating process in groups often depends on a choice of group generating systems, which is particularly true for the case of computerized calculations. An adequate choice of generating systems may essentially facilitate both calculations and the proof of theoretical facts. As such a system an ordinary base is used in finitely generated Abelian groups, the Maltsev base [1] in nilpotent groups and the Hall base [2] (see also [3, 4] in free groups. It is known [1,5,6] that the bases play an important role in the theory of finitely generated Abelian groups.

By constructing on an Abelian group A the Maltsev base, this group can be coordinated by means of vectors whose length is equal to some ordinal number λ that depends on A. Furthermore, the Maltsev base enables us to construct a triangular representation of a group A by means of generating and defining relations. One part (Sections 1 and 2) of this paper is dedicated to the exposition of these ideas, while in the other part (Sections 3 and 4) the following problem is solved. Given two Abelian groups and their respective Maltsev bases and triangular representations, it is required to construct by this information a Maltsev base and a triangular representation for their tensor product. The results obtained in this paper are used to prove the theorems on nilpotent groups of nilpotency class ≤ 3. They are surely also helpful for generalizing these theorems to nilpotent groups whose nilpotency class is more than 3.

Maltsev bases and triangular representations are nowadays widely used for calculations in polycyclic groups. In that case, bases consist of a finite number of elements, while representations contain a finite number of relations.

It should be noted that V. Bludov’s work [7] deals with a more general notion of a base, namely, with the so-called fiber base for groups. The present paper shares some ideas with V. Bludov’s work.

All definitions and results on Abelian groups used in this paper can be found in the monograph [8].

1. Triangular Representations of Periodic Groups

Let A be an Abelian group.

Definition 1.1. The set $T = \{t_i | \ i \in I\}$ is called *the Maltsev base* of the group *A* if

1) the set *I* is completely ordered and it is assumed that its ordinal type is an ordinal number ρ ;

2) for any ordinal number λ , $1 \le \lambda \le \rho$ we denote $A_\lambda = \langle t_i | \ i < \lambda \rangle$. Then $t_{\lambda+1} \notin A_\lambda$. A_λ is a normal subgroup of $A_{\lambda+1}$;

3) The group *A* is generated by the set *T*.

The *reduced order* $\bar{o}(t_{\lambda+1})$ of an element $t_{\lambda+1}$ is defined as follows: the reduced order is equal to a natural number *m* if $t_{\lambda+1}^m \in A_\lambda$ and *m* is the smallest natural number with this property, and is equal to ∞ otherwise.

From the definition of a Maltsev base it is obvious that

$$A_1 < A_2 < \cdots < A_\lambda < A_{\lambda+1} < \cdots \tag{1.1}$$

is a strictly increasing chain of subgroups and $A_\rho = A$. A factor group $A_{\lambda+1}/A_\lambda$ is a cyclic group of order $\bar{o}(t_{\lambda+1})$. Therefore any element *x* from *A* can be uniquely written in the form

$$x = t_{i_1}^{\alpha_{i_1}} t_{i_2}^{\alpha_{i_2}} \cdots t_{i_k}^{\alpha_{i_k}} , \tag{1.2}$$

where $i_1 < i_2 < \cdots < i_k$ and $\alpha_{i_k} \in Z$ if the reduced order of an element t_{i_j} is equal to infinity, and $0 \le \alpha_{i_j} < m_{i_j}$ if $m_{i_j} = \bar{o}\left(t_{i_j}\right)$.

Let us introduce the notion of *a triangular representation* associated with the Maltsev base *T*. Suppose an element $t_{\lambda+1}$ has a finite reduced order $m_{\lambda+1}$. Then, by definition, $t_{\lambda+1}^{m_{\lambda+1}} \in A_\lambda$ and therefore in view of (1.2)

$$t_{\lambda+1}^{m_{\lambda+1}} = t_{i_1}^{\alpha_{i_1}} t_{i_2}^{\alpha_{i_2}} \cdots t_{i_k}^{\alpha_{i_k}} \tag{1.3}$$

and $i_1 < i_2 < \cdots < i_k < \lambda + 1$.

Denote by *R* the set of equalities of form (1.3) for all base elements with finite reduced orders. Then by induction with respect to ordinal numbers one can easily prove

Proposition 1.2. *A group A on the set of generating T's has the representation*

$$A = \langle T \mid R \rangle .$$

Proof. It is obvious that all relations from *R* are fulfilled on the group *A*. On the other hand, if the group has the representation $\langle T \mid R \rangle$, then each of its elements can be written in form (1.2). Hence the proof follows.

Definition 1.3. A representation of *A* in the form $A = \langle T \mid R \rangle$ as described above is called a triangular representation of the group *A* associated with the base *T*.

Let the group $A = A_1 \oplus A_2$ be a direct sum of the groups A_1 and A_2 for which the Maltsev bases T_1, T_2 and systems of triangular relations R_1, R_2 are known. We put $T = T_1 \cup T_2$ and $R = R_1 \cup R_2$. Let us perform a complete ordering of the set *T* as follows: any element from T_1 precedes any element from T_2, while inside these elements the previous order is retained. Then it can be immediately verified that the set *T* is the Maltsev base for the group *A*, the reduced orders of elements from *A* retain their initial values and *R* is the system of triangular relations for *A*. How to generalize this construction in terms of a direct sum for any number of summands is obvious.

This reasoning proves

Proposition 1.4. *Let* $A = \bigoplus_{i \in I} A_i$ *, and* T_i*,* R_i *be given for any* $i \in I$ *, where I is a complete ordered set of indices. Then for defining the base and the system of triangular relations for the group A there exists a canonical procedure as described above.*

Let *B* be a subgroup of *A* and let $T(B)$, $R(B)$ and also $T(A/B)$, $R(A/B)$ be already constructed. Using this information, we will construct the Maltsev base and the system of triangular relations for the group *A*.

Proposition 1.5. *Let* $0 \to B \to A \to A/B \to 0$ *be a short exact sequence of Abelian groups. If the Maltsev bases and respective triangular repsentations are assumed to be given for the groups B and* A/B*, then there exists a canonical procedure for defining the base and the system of triangular representations for the group A.*

Proof. For any element $t \in T(A/B)$ we choose its some pre-image t' in the group *A* and denote by T' the set of all such pre-images for elements from $T(A/B)$. We assume $T = T(B) \cup T'$ and perform the ordering of elements from *T* so that elements from $T(B)$ would precede elements from T', while inside these sets we retain the initial orders. Then it is clear that the resulting system is a Maltsev base, and its elements keep the initial reduced orders.

Let us construct the system of triangular relations for *A*. For this, we use all relations $R(B)$ unchanged, while in relations from $R(A/B)$ we replace each base letter *t* with its pre-image t'. The left-hand part of a relation from $R(A/B)$ will differ from the right-hand part by a multiplier from a subgroup *B*. We multiply the right-hand part by this element from *B* and write it in terms of the base of the subgroup *B*. The obtained relations will be the sought ones.

Let *A* be a periodic group. Then *A* is a direct sum of its primary components A_p. By virtue of Proposition 1.4 it is sufficient to construct the Maltsev bases only for the primary components A_p. So, let *A* be a *p*-group. Let us recall the definition of a base subgroup for a *p*-group. A base subgroup *B* of the periodic *p*-group *A* is defined by the following conditions:

1) *B* is a direct sum of cyclic groups whose orders are powers of the prime number *p*;
2) *B* is a serving subgroup of the group *A*;
3) A/B is a divisible group.

In [7] it is proved that for any *p*-group *A* there always exists a base group *B*. Due to Propositions 1.4 and 1.5 it is sufficient to define the Maltsev bases and the system of triangular relations for the base subgroup *B* and the divisible group A/B.

Since the group *B* is a direct sum of cyclic groups whose orders are powers of the prime number *p*, by Proposition 1.5 it is sufficient to calculate the base and the system of triangular relations for the primary cyclic group, which can be done in a natural manner. The divisible group A/B is a direct sum of quasicyclic groups with respect to *p*. Therefore, by Proposition 1.4, it is sufficient to find the base and the system of triangular relations for a quasicyclic group $\mathbb{C}_{p^\infty}$. We choose in it an element *a* of order *p*. Then the set of elements

$$t_1 = a,\ t_2 = \frac{a}{p},\ \ldots,\ t_n = \frac{a}{p^n}, \ldots$$

is the Maltsev base for $\mathbb{C}_{p^\infty}$, and

$$t_{n+1}^p = t_n, \quad n = 1, 2, \ldots$$

is the system of triangular relations.

2. Auxiliary Results on Tensor Products

All definitions and results on tensor products of Abelian groups used in this paper can be found in the monograph [7, Ch. 10]. Let us recall some facts needed for our discussion.

1) If either a group *A* or a group *C* is *p*-divisible (divisible), then $A \otimes C$ is a *p*-divisible (divisible) group.

2) The inequality

$$h_p(a \otimes c) \ge h_p(a) + h_p(c)$$

is valid for the heights of elements.

3) If either a group *A* or a group *C* is a *p*-group (a periodic group), then $A \otimes C$ is a *p*-group (a periodic group).

4) If *A* is a *p*-divisble group and *C* is a *p*-group, then $A \otimes C = 0$. In particular $A \otimes C = 0$ if *A* is a *p*-group and *C* is a *q*-group, where *p*, *q* are different prime numbers. Moreover, $A \otimes C = 0$ if *A* is a divisible and *C* a periodic group.

5) If the inclusions $a \in mA$ and $c \in C[m]$ hold for some $m \in$, then $a \otimes c = 0$ in the group $A \otimes C$.

6) If $h_p(a)=\infty$ and C is a p-group, then $a\otimes c=0$ in the group $A\otimes C$ for any element $c\in C$.

7) There exists a natural isomorphism

$$\otimes C \cong C .$$

8) For any integer number m there exists a natural isomorphism

$$(m)\otimes C \cong C/mC .$$

In particular $(p^r)\otimes \ (p^s)\cong \ (p^t)$ where $t=\min(r,s)$.

9) Let the groups A and C be direct sums, $A=\underset{i\in I}{\oplus} A_i$, $C=\underset{j\in J}{\oplus} C_j$. Then $A\otimes C\cong \underset{i,j}{\oplus}(A_i\otimes C_j)$.

Theorem 2.1. *If*

$$0\longrightarrow A\xrightarrow{f}B\xrightarrow{g}C\longrightarrow 0$$

is a serving exact sequence, then for any group G the sequence

$$0\longrightarrow A\otimes G\xrightarrow{f^*}B\otimes G\xrightarrow{g^*}C\otimes G\longrightarrow 0$$

is a serving exact one.

Theorem 2.2. *If C is a p-group and B is a p-base subgroup of the group A, then we have a natural isomorphism* $A\otimes C\cong B\otimes C$.

Theorem 2.3. *For any groups A, C we have the isomorphisms*

$$T(A\otimes C)\cong[T(A)\otimes T(C)]\oplus[T(A)\otimes C/T(C)]\oplus[A/T(A)\otimes T(C)],$$
$$(A\otimes C)/T(A\otimes C)\cong A/T(A)\otimes C/T(C).$$

3. Triangular Representations of Tensor Products of *S*-Groups

We distinguish a wide class of Abelian groups for which the structure of tensor products can be calculated exactly.

Definition 3.1. An Abelian group A is called a S-group if $A/T(A)$ is decomposed into a direct sum of groups everyone of which is or .

Remark 3.2. As is known (see [7]), for any S-group A its periodic part is a direct summand. Hence A is representable in the form

$$A=T(A)\oplus F(A)\oplus D(A),$$

where $F(A)$ is a free Abelian group and $D(A)$ is a divisible group.

Let $T_1=\{t_i|\ i\in I_1\}$ be a Maltsev base of the group $T(A)$ and R_1 be the corresponding triangular system of relations. Let further $T_2=\{x_i|\ i\in I_2\}$ be the base for $F(A)$ and $T_3=\{y_i|\ i\in I_3\}$ be the base for $D(A)$.

In this situation the corresponding systems R_2 and R_3 of triangular relations are empty sets. Therefore we denote R_1 by R_A. Let us take another S-group $A'=T(A')\oplus F(A')\oplus D(A')$ with the respective bases $T_1'=\{t_i'|\ i\in I_1'\}$, $T_2'=\{x_i'|\ i\in I_2'\}$, $T_3'=\{y_i'|\ i\in I_3'\}$ and the corresponding system of triangular relations $R_{A'}$.

Our main task in this paper is to construct by means of the above bases and triangular relations the base and the system of triangular relations for the tensor product $A\otimes A'$.

Using Theorem 2.3 and the properties of tensor products, it is not difficult to prove that $A\otimes A'$ is a S-group and the following natural isomorphisms are valid:

$$T(A\otimes A')\cong[T(A)\otimes T(A')]\oplus[T(A)\otimes F(A')]\oplus[T(A')\otimes F(A)], \tag{3.1}$$

$$F(A\otimes A')\cong F(A)\otimes F(A'), \tag{3.2}$$

$$D(A\otimes A')=[F(A)\otimes D(A')]\oplus[F(A')\otimes D(A)]\oplus[D(A)\otimes D(A')]. \tag{3.3}$$

Let us calculate the bases for $F(A\otimes A')$ and $D(A\otimes A')$. The above-mentioned properties of tensor products immediately imply that

$$\overline{T}_2=\{(x_i\otimes x_j')|\,(i,j)\in i_2\times I_2'\},$$
$$\overline{T}_3=\{(x_i\otimes y_j')|\,(i,j)\in i_2\times I_3';(x_i'\otimes y_j)\,|\,(i,j)\in I_2'\times I_3;(y_i\otimes y_j')|\,(i,j)\in I_3\times I_3'\}.$$

Thus it remains to find the base and the system of triangular relations for the group $T(A\otimes A')$. Since the latter group consists of three summands, we need to find the base and the system of triangular relations for each summand individually. Let us begin with the group $T(A)\otimes T(A')$. By Theorem 2.1 we have

$$T(A)\otimes T(A')\cong \underset{p}{\oplus}\left(B_p\otimes B'_p\right),$$

where B_p and B'_p are the respective base subgroups in *p*-primary components. $B_p\otimes B'_p$ is a direct sum of cyclic groups. If T_p and T'_p are the bases of the groups B_p and B'_p, respectively, then the base for $B_p\otimes B'_p$ is $T_p\otimes T'_p$, while the triangular representation of the group $B_p\otimes B'_p$ consists of $(t\otimes s)^m=0$, where $t\in T_p$, $s\in T'_p$ and $m=\text{g.c.d.}\left(o(t),o(s)\right)$ – greatest common divisor.

Two other summands are symmetrical. Therefore it is sufficient to find the base and the corresponding system of triangular relations for one of them, say, for $T(A)\otimes F(A')$. Since $F(A')$ is a free Abelian group with the base $T'_2=\left\{x'_i \mid i\in I'_2\right\}$, we have

$$T(A)\otimes F(A')\cong \underset{i\in I'_2}{\oplus}\left(T(A)\otimes\langle x'_i\rangle\right)\cong \underset{i\in I'_2}{\oplus}\left(T(A)\right)_i,$$

where $\left(T(A)\right)_i\cong T(A)$, $i\in I'_2$.

But the base and the corresponding system of triangular relations for the group $T(A)$ are known. We recall that they are parts of the corresponding sets for the group *A*. Therefore the construction of the base and the corresponding system of triangular relations for the group $A\otimes A'$ is completed.

4. Triangular Representations of Torsion-Free Abelian Groups

4.1 The case of reduced groups. Let *A* be a torsion-free Abelian group. Then

$$A\cong A_{red}\oplus A_d\ . \tag{4.1}$$

If A' is another torsion-free Abelian group, then

$$\left(A\otimes A'\right)_{red}\cong A_{red}\otimes A'_{red}\,, \tag{4.2}$$

$$\left(A\otimes A'\right)_d\cong\left(A_{red}\otimes A'_d\right)\oplus\left(A'_{red}\otimes A_d\right)\oplus\left(A_d\otimes A'_d\right). \tag{4.3}$$

Since the divisible part is a vector space over Θ, its Maltsev base can be immediately calculated by maximal, linearly independent systems of elements of the groups A_{red}, A'_{red} and the bases of the groups A_d, A'_d (this is a tensor product of the respective bases for the multipliers in formula (9)). In this case, the triangular system of relations is empty. Therefore it is sufficient to construct the triangular system of relations only for the reduced torsion-free groups *A* and A'.

4.2 The case of Abelian groups decomposed into a direct sum of groups of rank 1. Let

$$A=\underset{i\in I}{\oplus}B_i,\ A'=\underset{i\in I'}{\oplus}B'_i\,,$$

where B_i and B'_i are groups of rank 1. Then the tensor product $A\otimes A'$ takes the form

$$A\otimes A'\cong\underset{i,j}{\oplus}\left(B_i\otimes B_j\right).$$

In that case, by virtue of 4.1, the construction of the system of triangular relations of the tensor product $A\otimes A'$ reduces to the construction of the system of triangular relations for the tensor products of groups of rank 1. So, let *A* and A' be groups of rank 1. We choose the nonzero elements a, a' in the groups *A*, A', respectively, and denote by $\chi(a)$, $\chi(a')$ the characteristics of the elements a, a'. On the set $N\cup\{\infty\}$ we define the operation of addition as follows: as addition of natural numbers for elements from *N*, and $n+\infty=\infty+n=\infty$ and $\infty+\infty=\infty$. Then, by the property of tensor products, for the heights of elements (see Section 2) the following formula is valid:

$$h_p\left(a\otimes a'\right)\geq h_p(a)+h_p(a'),$$

which in the case of torsion-free Abelian groups becomes an equality.

Therefore

$$\chi(a \otimes a') = \chi(a) + \chi(a'),$$

where addition is performed componentwise. Also, if the characteristic $\chi(a \otimes a')$ consists entirely of infinity, then $A \otimes A' \cong \mathbb{Q}$ and for it the triangular system of relations is empty. Otherwise we denote by A_0 the group generated by the tensor $a \otimes a'$. Then the factor group $(A \otimes A')/(a \otimes a')$ is a periodic group. Its primary p-component is a cyclic group of order p^k if $h_p(a \otimes a') = k$, and is a quasicyclic group if $h_p(a \otimes a') = \infty$. In Subsection 4.1 it is explained how in that case we can construct the base for the group $A \otimes A'$ and the corresponding systems of triangular relations.

4.3. The general case. Let us first consider a subcase where A is an arbitrary reduced group, and A' is a reduced group of rank 1. We choose in the group A a maximal, linearly independent system $M = \{x_i \mid i \in I\}$ and perform the complete ordering of the set of indices I. Let A_1 denote the serving closure of the element x_1 in A. Then A_1 is a group of rank 1, and the factor group A/A_1 is torsion-free. Then, by virtue of Theorem 2.1, the following exact sequence is valid:

$$0 \longrightarrow A_1 \otimes A' \longrightarrow A \otimes A' \longrightarrow A/A_1 \otimes A' \longrightarrow 0\,.$$

The system of triangular relations for $A_1 \otimes A'$ has already been found since they both are groups of rank 1. By virtue of Proposition 1.5, to construct the system of triangular relations it is sufficient to find the corresponding system of triangular relations for the group $A/A_1 \otimes A'$. Further we use transfinite induction with respect to the ordinal type ρ of the set of indices I. Let us perform one more step of this induction. Denote the factor group A/A_1 by $\overline{A}$, and $\overline{x}_2, \overline{x}_3, \ldots$ the images of the respective elements from M.

Let $\overline{M} = \{\overline{x}_2, \overline{x}_3, \ldots\}$ and $\overline{A}_1$ be the serving closure of an element $\overline{x}_2$ in $\overline{A}$. Applying Theorem 2.2 to the pair of groups $\overline{A}_1$, $\overline{A}$, we construct the system of triangular relations for the group $\overline{A}_1 \otimes \overline{A}'$. Now the problem of construction of the base reduces to the problem of construction for the group $\overline{A}/\overline{A}_1 \otimes A'$. Transfinite induction completes the proof.

Now let us consider the general case where the groups A and A' are not groups of rank 1. We choose a maximal, linearly independent system $M' = \{x_i' \mid i \in I'\}$ and perform a complete ordering of the set I'. Denote by A_1' the serving closure of the element x_1'. Then by virtue of Theorem 2.1 we have

$$0 \longrightarrow A \otimes A_1' \longrightarrow A \otimes A' \longrightarrow A \otimes A'/A_1' \longrightarrow 0\,.$$

Since A_1' is a group of rank 1, the triangular system of relations has already been constructed for it in Subsection 4.2. Therefore it is sufficient to construct the triangular system of relations for the group $A \otimes A'/A_1'$. For this we use induction with respect to the ordinal type of the set of indices I_1'.

4.4. The tensor product of arbitrary Abelian groups. Let A and A' be arbitrary Abelian groups. Then by Theorem 3.3 we have

$$T(A \otimes A') \cong [T(A) \otimes T(A')] \oplus [T(A) \otimes A'/T(A')] \oplus [A/T(A) \otimes T(A')]. \tag{4.4}$$

We have already constructed the system of triangular relations for the group $A/T(A) \otimes A'/T(A')$. Therefore it is sufficient to construct such a system for the group $T(A \otimes A')$. By virtue of isomorphism (4.4) it suffices to construct the system of triangular relations for the tensor product of the groups $T(A) \otimes T(A')$, $T(A) \otimes A'/T(A')$, $A/T(A) \otimes T(A')$. Since periodic groups are S-groups, the triangular system of relations of the group $T(A) \otimes T(A')$ has already been constructed in Subsection 4.3. Since the last two groups are symmetrical, it is sufficient to construct the system of triangular relations for one of them, say, for $T(A) \otimes A'/T(A')$. By Theorem 2.2 a p-primary component is isomorphic to the group $T(A_p) \otimes B_p$, where B_p is a p-base subgroup $A'/T(A')$. Since $T(A_p)$ and B_p are S-groups, we have already constructed the system of triangular relations for them in Subsection 4.3.

მათემატიკა

აბელური ჯგუფების ტენზორული ნამრავლების მალცევის ბაზისები და სამკუთხა წარმოდგენები

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