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BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 4, no. 1, 2010

Mathematics

The Lattice of Fully Invariant Submodules of a Reduced Cotorsion *p*-adic Module

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(Presented by Academy Member H. Inassaridze)

ABSTRACT. The paper considers the lattice of fully invariant submodules of a reduced cotorsion *p*-adic module $T^{\bullet} \oplus C$, where *T* is a countable direct sum of torsion-complete *p*-groups, and *C* is a torsion-free, algebraically compact group. It is shown that this lattice is isomorphic to the lattice of filters of a semilattice made up of infinite matrices and indicators. *© 2010 Bull. Georg. Natl. Acad. Sci.*

Key words: *cotorsion group, torsion-complete p-group, fully invariant submodule.*

The paper deals with questions of the Abelian group theory and the term *group* always means an additively written Abelian group. The notation and terms used in the paper are taken from [1, 2].

We denote by p a fixed prime number. Z and Q are respectively the group of integers and the group of rational numbers. In the *p*-adic topology, the completion of a torsion p-group T is denoted by the symbol $\hat{T}$. Its torsion part $\bar{T}$ is a torsion-complete group. A subgroup B of a group A is called fully invariant if every endomorphism of A maps B into B. Such are, for example, subgroups

$$nA = \{na \mid a \in A\}, \quad A[n] = \{a \mid na = 0, a \in A\}, \; n \geq 0, \; n \in Z,$$

or the torsion part of a group A.

The investigation of the lattice of fully invariant subgroups of a group is an important task of the theory of Abelian groups. For a sufficiently wide class of p-groups this topic was studied by R. Baer [3], I. Kaplansky [4], P. Linton [5], R.Pierce [6], J. Moore and E. Hewett [7] and other authors. For torsion-free and mixed groups, problems related to this topic were investigated in the works of A. Mader [8], R. Göbel [9], A. Moskalenko [10], S. Grinshpon, P. Krylov [11], V. Misyakov [12] and other authors.

Little is known about the lattice of fully invariant subgroups in the class of cotorsion groups. A group A is called a cotorsion group if its extension by any torsion-free group C splits, i.e. $Ext(C, A) = 0$. The importance of the class of cotorsion groups rests on the following two facts: for any groups A and B the group $Ext(A, B)$ is a cotorsion group; any reduced group A is isomorphically embeddable into the group $A^{\bullet} = Ext(Q/Z, A)$ called the cotorsion hull of A; also, $A^{\bullet}/A$ is a torsion–free divisible group. Any reduced cotorsion group A can be represented as a direct sum $A = T^{\bullet} \oplus C$, where $T^{\bullet} \cong Ext(Q/Z, T)$, $T = tA$ is the torsion part of the group A, while $C \cong Ext(Q/Z, A/T)$ is a torsion-free, algebraically compact group. If the group T is written as a direct sum of primary components $T=\oplus_p T_p$, then we have $Ext(Q/Z,T)\cong\prod_p Ext\left(Z\left(p^\infty\right),T_p\right)=\prod_p T_p^\bullet$, $Ext(Q/Z,A/T)\cong\prod_p Ext\left(Z\left(p^\infty\right),A/T\right)=\prod_p C_p$ and

$$A=\prod_p (T_p^\bullet\oplus C_p).$$

Thus the study of reduced cotorsion groups is reduced to a considerable extent to the study of groups of the form

$$A=T^\bullet\oplus C\,, \tag{1}$$

where T is a p-group, and C is a torsion-free, algebraically compact group. As it is shown in [1, §§40, 54], $T^\bullet, C$ and therefore A too, are p-adic modules.

When studying the lattice of fully invariant subgroups of a group A, active use is made of the notions of an element indicator and a fully transitive group.

By *the p-indicator* of an element a of a group A we mean the increasing sequence of ordinal numbers

$$H_A(a)\equiv H(a)=(h(a),h(pa),...h(p^n a),...),$$

where h denotes the generalized p-height of a, i.e., $h(a)=\sigma$ if $a\in p^\sigma A\setminus p^{\sigma+1}A$ and $h(0)=\infty$ (of course, if $h(p^n a)=h(0)=\infty$, then $h(p^{n+1}a)=0$). In a set of indicators we can introduce the order

$$H(a)\le H(b)\Leftrightarrow h(p^i a)\le h\left(p^i b\right),\quad i=0,1,...$$

A reduced p-group is called *fully transitive* if, assuming that $H(a)\le H(b)$, for arbitrary elements a and b there exists an endomorphism φ of the group such that $\varphi a=b.$ In fully transitive groups, the lattice of fully invariant subgroups is studied with the aid of indicators. In particular, I. Kaplansky showed that such a subgroup has the form

$$A(u)=\{a\in A \mid H(a)\ge u\},$$

where $u=(\sigma_o,\sigma_1,...\sigma_n,...)$ is an increasing sequence of ordinal numbers and symbols ∞ that satisfies the following condition: if between σ_n and σ_{n+1} there is a jump, then in A there are an element of order p and height σ_n (see [2, Theorem 67.1]).

A. Mader [8] showed that an algebraically compact group is fully transitive and described by means of indicators of the lattice of fully invariant subgroups of an algebraically compact group. He also indicated the generalized conditions, the fulfillment of which gives a description of the lattice of fully invariant submodules.

Theorem 1. (A. Mader). *Let A be a module over a commutative ring R, Δ be the lattice of all its fully invariant submodules, Ω be some lower semilattice, and $\Phi: A\to\Omega$: be the map with the following properties:*

1) Φ *is surjective;*

2) $\Phi(fa)\ge\Phi(a)$ $\forall a\in A$ *and* $f\in EndA$;

3) $\Phi(a+b)\ge\Phi(a)\wedge\Phi(b)$;

4) *if* $\Phi(a)\ge\Phi(b)$, *then there exists an endomorphism f of the module A such that* $f(b)=a$;

5) *if* $C\in\Delta$, *then for any* $a,b\in C$ *there exists* $c\in C$ *such that* $\Phi(c)=\Phi(a)\wedge\Phi(b)$.

Then the set Ω^* of all filters of Ω, which is ordered with respect to the inclusion, is a lattice and the mapping $\alpha:\Omega^*\to\Delta$ defined by the rule $\alpha(D)=\{\alpha\in A \mid \Phi(a)\in D\}$ is a lattice isomorphism.

Like in the case of p-groups, we will formulate the notion of full transitivity in the group $T^\bullet=Ext(Z(p^\infty),T)$. If T is a torsion-complete group, then its cotorsion hull is an algebraically compact group (see [1, §56]) and, as mentioned above, is fully transitive. A. Moskalenko [10] proved that when T is a direct sum of cyclic p-groups, then $T^\bullet$ is also fully transitive and all the conditions of Theorem 1 are fulfilled. Therefore in this case, too, the lattice Ω^* of filters of indicators describes the lattice of fully invariant subgroups. The direct sum of torsion-complete groups is a natural generalization of direct sums of cyclic p-groups and torsion-complete groups. As the author has shown in [13], in this class of groups the cotorsion hull is not fully transitive if the sum is infinite. Hence, because of condition 4) of Theorem 1 the lattice of fully invariant subgroups cannot be described by means of indicators.

In [14] the lattice of fully invariant subgroups of the group $T^{\bullet}$ is studied in the case, in which T is a countable direct sum of torsion complete p-groups:

$$T = \overset{\infty}{\underset{j=1}{\oplus}} \overline{B_j}\,, \tag{2}$$

where B_j is a basic subgroup of $\overline{B_j}$, and $B = \bigoplus_{j=1}^{\infty} B_j$ is a basic subgroup of T. When T is a separable p-group, elements of the cotorsion hull $T^{\bullet}$ are represented as countable sequences (see [10])

$$T^{\bullet} = \{(a_o, a_1 + T, ... a_i + T, ...) \mid a_i \in \hat{T}, \quad pa_{i+1} - a_i \in T,\ i = 0,1...\}.$$

Using this representation of elements, we can easily calculate their height and indicator.

Let $B = \underset{a\in J}{\oplus}\langle x_a \rangle$ be some fixed basic subgroup of the separable p-group T. If $a \in T^{\bullet}$, $a = (a_0, a_1 + T, ...)$, then the group B contains a sequence (b_i), $i = 0,1,...,$ such that for any i

$$b_i = \sum_{j=1}^{s} m_j x_{a\,j}, \qquad 0 \le m_j < p, \quad \mathit{and} \quad a_i = \lim_{n\to\infty}\left(\sum_{s=0}^{n} p^s b_{i+s}\right). \tag{3}$$

This representation of an element a is called *canonical*. We say that the sequence (b_i) corresponds to the canonical representation of an element a.

Let a group T have form (2) and $a \in T^{\bullet}$ (see (3). Denote by π_i the projection of the group T on the direct summand $\overline{B_i}$ and consider the sequence

$$\pi_i(b_j) = (b_{i\,j}), \quad j = 0,1,....$$

For every $i \ge 1$ and fixed *j*, the sequence $b_{i0}, b_{i1},...$, defines the element $a_{i\,j} = \lim_{n\to\infty}\sum_{s=0}^{n} p^s b_{i\,j+s}$, while the elements $a_{i0}, a_{i1},...$ of the group $\hat{B}_i$ define the element $a^{(i)} = (a_{i0}, a_{i1} + T,...)$ of the group $T^{\bullet}$. It is obvious that

$$a_j = \lim_{n\to\infty}\sum_{i=1}^{n} a_{i\,j}.$$

Note that the elements $a^{(i)}, i = 1,2,...$ are uniquely defined by the element a from (3) since if $(b'_j)_j \ge 0$ is another sequence corresponding to the element a and $(a_0, a_1 + T,...) = (a'_0, a'_1 + T,...)$, then $a_0 = a'_0$ (see [10, §§1, 2]), and if $a_k - a'_k \in T$, then $\hat{\pi}_i(a_k) = a_{i\,k}$, $\hat{\pi}_k(a'_k) = a'_{i\,k}$ and $a_{ik} - a'_{ik} = \hat{\pi}_i(a_k - a'_k) \in T, k = 1,2,...$, where $\hat{\pi}_i$ denotes the induced projection on the group $\hat{B}_i$.

To each element $a \in T^{\bullet}$ we assign the matrix

$$\Phi\ (a) = \|H(a_{i0})\|,\ i \ge 0,$$

where $H(a_{00}) = H_{T^{\bullet}}(a)$, $H(a_{i0}) = H_{\hat{T}}(a_{i0})$ for $i \ge 1$.

Definition 1. The matrix $\|k_{ij}\|_{i,j\ge 0}$ made up of ordinal numbers and symbols ∞ is called *admissible* with respect to the group T if the following conditions are fulfilled:

1. The first row is an increasing sequence of ordinal numbers smaller than $\omega + \omega$ or of symbols ∞, and if $k_{0j} \ge \omega$, , then $k_{on+1} = k_{0n} + 1$ for any $n \ge j$. The other rows are increasing sequences of non-negative integers or symbols ∞ (it is assumed that $\infty + 1 = \infty$).

2. If $k_{on} = \omega + m$ in the first row is the first infinite ordinal number and $m < n$, then infinitely many rows contain a non-negative integer and there exists a row i_0 such that $k_{in-m} = \infty$ for $i \ge i_0$. If $k_{0n} = \omega + m,\ m \ge n,$ then, starting from some i_0, all rows consist only of symbols ∞.

3. If all elements in a row are non-negative integers, then this row contains infinitely many jumps.

4. If between $k_{ij\,j}$ and k_{ij+1} there is a jump, then in the group B_i there exists a base element of order $p^{k_{ij}+1}$ (it is assumed that $B_0 = B$).

5. In every column $k_{ij} \to \infty$ as $i \to \infty$; also, if $k_{0j} \neq \omega + m,$ then $k_{oj} = \min\{k_{1j}, k_{2j}, ...\},$ and if $k_{0j} = \omega + m,$ then $k_{1j} = k_{2j} = ... = \infty$.

We denote the set of admissible matrices (with respect to T) by Ω and consider in it the reflexive and transitive relation $\leq$ (see [13], Definition 1.2). Then the relation

$$U\rho V \Leftrightarrow [\bigcup \leq V \; and \; V \leq U]$$

is the equivalence on the set Ω , and the relation $\leq$

$$\overline{U} \leq \overline{V} \Leftrightarrow U \leq V$$

defined on the quotient set $\Omega / \rho = \overline{\Omega}$ is the order.

In [14] it is shown that the function $\overline{\Phi} : T^\bullet \to \overline{\Omega}, \quad \overline{\Phi}(a) = \overline{\Phi(a)}$ defined on the group $T^\bullet$, where T has form (2) and Ω is the set of admissible matrices with respect to T, satisfies all the conditions of Theorem 1. Therefore the following statement is true.

Theorem 2. *The lattice of fully invariant subgroups of the cotorsion hull* $T^\bullet$, *where* T *is a countable direct sum of torsion-complete* p*-groups, is isomorphic to the lattice of filters of the semilattice* $\overline{\Omega}$.

Let us return to the reduced cotorsion p-adic module (1), where T has form (2), and investigate the lattice of fully invariant submodules of the module A . To this end, we consider the set

$$\Omega^\bullet = \overline{\Omega} \bigcup \overline{H} \, ,$$

where $\overline{\Omega}$ is the semilattice figuring in the formulation of the Theorem 2 and $\overline{H}$ is the set of increasing sequences of nonnegative integers, containing only a finite number of jumps. If between k_i and k_{i+1} there is a jump, then in the basic subgroup B of the group T there exists a base element of order p^{k_i+1} .

The elements of the set $\overline{\Omega}$ are the classes of admissible matrices, where the first row is uniquely defined. Let us define the relation $\leq$ on the set $\Omega^\bullet$. If $\overline{K}$ and $\overline{K}' \in \overline{\Omega}$, then $\overline{K} \leq \overline{K}'$ is defined in the same manner as in the case of the set $\overline{\Omega}$. Otherwise, it is required that the following condition should be fulfilled: $k_i \leq k_i' \quad i = 0,1,...$ where $(k_i), (k_i'), i \geq 0$ are the first rows of matrix taken from the classes of the set $\overline{\Omega}$ or sequences from $\overline{H}$. It is obvious that the relation $\leq$ is a relation of order on the set $\Omega^\bullet$.

If $\overline{K}, \overline{K}' \in \Omega$, then the exact lower bound $\inf\left(\overline{K}, \overline{K}'\right)$ of these elements is defined in the same way as this was done for the set of $\overline{\Omega}$. If $\overline{K} \in \overline{\Omega}$ and $H = (k_o', k_1', ...) \in \overline{H}$ are assumed to be given, then we have inf($\overline{K}, H) = \left(\min\left(k_{0j}, k_j'\right)\right)_{j\geq 0}$, where $(k_{0j})_{\,j \geq 0}$ is the first row of an admissible matrix K . Taking into account the definition of a matrix K , we obtain inf $(\overline{K}, H) \in \overline{H}$. Analogously, we define $\inf(H_1, H_2)$ for $H_1, H_2 \in \overline{H}$. It is easy to verify that this definition satisfies the requirements for an exact lower bound. Hence the set $\Omega^\bullet$ is the lower semilattice.

Let us consider the map

$$\Phi^\bullet : A \to \Omega^\bullet$$

which to each element $a \in T^\bullet$ assigns $\overline{\Phi}(a)$ or, if $a = t + c$, where $c \neq 0, \;\; c \in C$, then $\Phi^\bullet(a) = H(a),$ where $H(a)$ is the p-indicator of an element a. It is not difficult to verify that the function $\Phi^\bullet$ satisfies all the conditions of Theorem 1.

Thus the following statement is valid.

Theorem 3. *The lattice of fully invariant submodules of a reduced cotorsion* p*-adic module* A *is isomorphic to the lattice of filters of a semilattice* $\Omega^\bullet$.

Acknowledgement. This study was supported by grant ATSU/09316.

მათემატიკა

რედუცირებული კოგრეხვითი p-ადიური მოდულის სავსებით ინვარიანტულ ქვემოდულთა მესერი

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ნაშრომში განიხილება რედუცირებული კოგრეხვითი p-ადიური მოდულის სავსებით ინვარიანტულ ქვემოდულთა მესერი, როცა მოდულს აქვს $T^{\bullet} \oplus C$ სახე, სადაც $T^{\bullet}$ არის გრეხვითად სრული p-ჯგუფების თვლადი პირდაპირი ჯამი, ხოლო C — გრეხვისაგან თავისუფალი ალგებრულად კომპაქტური ჯგუფი. ნაჩვენებია, რომ აღნიშნული მესერი უსასრულო მატრიცებისა და ინდიკატორებისაგან აგებული ნახევარმესრის ფილტრთა მესრის იზომორფულია.

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Received October, 2009