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Mathematics

The Riemann Boundary Value Problem for Analytic Functions in the Frame of Grand $L^{p)}$ Spaces

Vakhtang Kokilashvili

Academy Member, A. Razmadze Mathematical Institute, Tbilisi

ABSTRACT. We solve the Riemann boundary value problem for analytic functions in the class of Cauchy-type integrals with the density in the grand Lebesgue $L^{p)}(\Gamma)$ spaces. We consider the case when a coefficient in boundary condition is everywhere nonvanishing continuous function and the right side function belongs to the same $L^{p)}$ space. The solvability conditions are established and the explicit formulas for solutions are given. *© 2010 Bull. Georg. Natl. Acad. Sci.*

Key words: *grand Lebesgue space, Carleson curve, the Riemann boundary value problem, Cauchy type integral.*

1. Introduction

Let Γ be an oriented rectifiable simple closed curve in the complex plane. We denote by D^+ and D^- the bounded and unbounded component of $\mathbb{C}\setminus\Gamma$, respectively.

The aim of the paper is to investigate the Riemann problem: find an analytic function Φ on the complex plane cut along Γ whose boundary values satisfy the conjugate condition

$$\Phi^+(t)=G(t)\Phi^-(t)+g(t), \qquad t\in\Gamma, \tag{1}$$

when G and g are given functions on Γ, and Φ^+ and Φ^- are boundary values of Φ on Γ from inside and outside Γ, respectively. This problem is also known as the problem of linear conjugation.

Problem (1) comes from Riemann [1]. Important results on which the posterior solution of problem (1) was based, were obtained by Yu. Sokhotski, D. Hilbert, I. Plemely and T. Carleman. A complete solution of the Riemann problem in the frame of Hölder continuous functions was given in the papers of Gakhov [2] and N. Muskhelishvili [3]. We refer also to the works [4-8] for investigation of the Riemann problem in classical L^p spaces.

Preliminaries. In the sequel we denote

$$D(t,r)=\Gamma\cap B(t,r), \qquad r>0$$

where $B(t,r)=\{z\in\mathbb{C}:\ |z-t|<r\}$.

A rectifiable curve Γ is called the Carleson curve, if there exists a constant $c_0>0$ not depending on t and r, such that

$$\nu D(t,r)\le c_0 r\ ,$$

where ν is the arc-length measure on Γ.

The grand Lebesgue space $L^{p)}(\Gamma)$ ($1<p<\infty$) is a Banach function space defined by the norm

$$\|f\|_{L^{p)}_w(\Gamma)} = \sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon}{\nu\Gamma}\int_\Gamma |f(t)|^{p-\varepsilon}\,d\nu\right)^{\frac{1}{p-\varepsilon}}. \tag{1}$$

The grand Lebesgue space $L^{p)}$ was introduced by T. Iwaniec and C. Sbordone [9].

Let

$$K^{p)}(\Gamma)=\left\{\Phi(z):\ \Phi(z)=\frac{1}{2\pi i}\int_\Gamma\frac{\varphi(\tau)d\tau}{\tau-z},\ z\notin\Gamma\ \text{with}\ \varphi\in L^{p)}(\Gamma)\right\}. \tag{2}$$

The mapping properties of the Hardy-Littlewood maximal functions and Hilbert transforms defined on finite interval in weighted $L^{p)}$ spaces were studied in [10] and [11] respectively. Recently we established the boundedness criterion in weighted $L^{p)}$ spaces ($1<p<\infty$) for Cauchy singular integrals and maximal functions defined on Carleson curves [12].

2. Main Result.

We proceed with the solution of the problem in the following setting: let Γ be Carleson curve. Let G be a continuous function on Γ with the condition $G(t)\neq 0$, $t\in\Gamma$. Let æ$=\frac{1}{2\pi}[\arg G(t)]_\Gamma$. Find an analytic function $\Phi\in K^{p)}(\Gamma)$, ($1<p<\infty$), satisfying the condition (1), where $g\in L^{p)}(\Gamma)$.

Theorem. *The following statements hold:*

i) for æ ≥ 0, *problem* (1) *is unconditionally solvable in the class* $K^{p)}(\Gamma)$ *and all its solutions are given by*

$$\Phi(z)=\frac{X(z)}{2\pi i}\int_\Gamma\frac{g(\tau)}{X^+(\tau)(\tau-z)}d\tau+X(z)Q_{æ-1}(z) \tag{3}$$

with

$$X(z)=\begin{cases}\exp h(\tau), & z\in D^+\\ (z-z_0)^{-æ}\exp h(z), & z\in D^-,\quad z_0\in D^+,\end{cases} \tag{4}$$

where $Q_{æ-1}$ *is an arbitrary polynomial of degree* æ-1 ($Q_{æ-1}(z)=0$);

ii) for æ<0, *problem* (1) *is solvable in the class* $K^{p)}(\Gamma)$ *if and only if*

$$\int_\Gamma\frac{g(t)t^k}{X^+(t)}dt=0,\quad k=0,1,\ldots,|æ|-1; \tag{5}$$

and under these conditions problem (1) *has the unique solution given by* (3) *with* $Q_{æ-1}$=0.

In forthcoming papers we will solve problem (1) in the case of oscillating coefficients G.

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მათემატიკა

რიმანის სასაზღვრო ამოცანა ანალიზური ფუნქციებისათვის $L^{p)}$ სივრცის ჩარჩოებში

ვ. კოკილაშვილი

აკადემიის წევრი, ა. რაზმაძის მათემატიკის ინსტიტუტი, თბილისი

ნაშრომში ამოხსნილია რიმანის სასაზღვრო ამოცანა ანალიზური ფუნქციებისათვის იმ კოშის ტიპის ინტეგრალით წარმოდგენად ფუნქციათა კლასებში, რომელთა სიმკვრივეები $L^{p)}$ ($1<p<\infty$) სივრცეებს მიეკუთვნებიან. ჩვენ განვიხილავთ იმ შემთხვევას, როცა სასაზღვრო პირობაში კოეფიციენტი ყველგან ნულისაგან განსხვავებული უწყვეტი ფუნქციაა და მარჯვენა მხარე $L^{p)}$ კლასის მოცემული ფუნქციაა. ჩვენ ვამტკიცებთ ამოხსნადობის პირობებს და ამონახსნებს ვწერთ ცხადი სახით.

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