The Riemann Boundary Value Problem for Analytic Functions in the Frame of Grand \( L^p \) Spaces

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ABSTRACT. We solve the Riemann boundary value problem for analytic functions in the class of Cauchy-type integrals with the density in the grand Lebesgue \( L^p(\Gamma) \) spaces. We consider the case when a coefficient in boundary condition is everywhere nonvanishing continuous function and the right side function belongs to the same \( L^p \) space. The solvability conditions are established and the explicit formulas for solutions are given. © 2010 Bull. Georg. Natl. Acad. Sci.

Key words: grand Lebesgue space, Carleson curve, the Riemann boundary value problem, Cauchy type integral.

1. Introduction

Let \( \Gamma \) be an oriented rectifiable simple closed curve in the complex plane. We denote by \( D^+ \) and \( D^- \) the bounded and unbounded component of \( \mathbb{C} \setminus \Gamma \), respectively.

The aim of the paper is to investigate the Riemann problem: find an analytic function \( \Phi \) on the complex plane cut along \( \Gamma \) whose boundary values satisfy the conjugate condition

\[
\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma. \tag{1}
\]

when \( G \) and \( g \) are given functions on \( \Gamma \), and \( \Phi^+ \) and \( \Phi^- \) are boundary values of \( \Phi \) on \( \Gamma \) from inside and outside \( \Gamma \), respectively. This problem is also known as the problem of linear conjugation.

Problem (1) comes from Riemann [1]. Important results on which the posterior solution of problem (1) was based, were obtained by Yu. Sokhotski, D. Hilbert, I. Plemely and T. Carleman. A complete solution of the Riemann problem in the frame of Hölder continuous functions was given in the papers of Gakhov [2] and N. Muskhelishvili [3]. We refer also to the works [4-8] for investigation of the Riemann problem in classical \( L^p \) spaces.

Preliminaries. In the sequel we denote

\[
D(t, r) = \Gamma \cap B(t, r), \quad r > 0
\]

where

\[
B(t, r) = \{ z \in \mathbb{C} : |z - t| < r \}.
\]

A rectifiable curve \( \Gamma \) is called the Carleson curve, if there exists a constant \( c_0 > 0 \) not depending on \( t \) and \( r \), such that

\[
\nu D(t, r) \leq c_0 r,
\]

where \( \nu \) is the arc-length measure on \( \Gamma \).
The grand Lebesgue space \( L^p(\Gamma) \) \((1 < p < \infty)\) is a Banach function space defined by the norm
\[
\|f\|_{L^p(\Gamma)} = \sup_{0<\varepsilon<\frac{1}{p-1}} \left( \frac{\varepsilon}{V(\Gamma)} \left\{ \int_{\Gamma} |f(t)|^{p-\varepsilon} \, dt \right\}^{\frac{1}{p-\varepsilon}} \right).
\] (1)

The grand Lebesgue space \( L^p \) was introduced by T. Iwaniec and C. Sbordone [9]. Let
\[
K^p(\Gamma) = \left\{ \Phi(z) : \Phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{g(\tau) \, d\tau}{\tau - z}, \, z \notin \Gamma \text{ with } \varphi \in L^p(\Gamma) \right\}.
\] (2)

The mapping properties of the Hardy-Littlewood maximal functions and Hilbert transforms defined on finite interval in weighted \( L^p \) spaces were studied in [10] and [11] respectively. Recently we established the boundedness criterion in weighted \( L^p \) spaces \((1 < p < \infty)\) for Cauchy singular integrals and maximal functions defined on Carleson curves [12].

2. Main Result.

We proceed with the solution of the problem in the following setting: let \( \Gamma \) be Carleson curve. Let \( G \) be a continuous function on \( \Gamma \) with the condition \( G(t) \neq 0, \, t \in \Gamma \). Let \( \varepsilon = \frac{1}{2\pi} \left[ \arg G(t) \right]_\Gamma \). Find an analytic function
\( \Phi \in K^p(\Gamma) \), \((1 < p < \infty)\), satisfying the condition (1), where \( g \in L^p(\Gamma) \).

**Theorem.** The following statements hold:

i) for \( \varepsilon \geq 0 \), problem (1) is unconditionally solvable in the class \( K^p(\Gamma) \) and all its solutions are given by
\[
\Phi(z) = \frac{X(z)}{2\pi i} \int_{\Gamma} \frac{g(\tau)}{X^+(\tau)(\tau - z)} \, d\tau + X(z)Q_{\varepsilon^{-1}}(z)
\] (3)

with
\[
X(z) = \begin{cases} 
\exp h(\tau), & z \in D^+ \\
((z-z_0)^{-\varepsilon} \exp h(z), & z \in D^-, \, z_0 \in D^+.
\end{cases}
\] (4)

where \( Q_{\varepsilon^{-1}} \) is an arbitrary polynomial of degree \( \varepsilon^{-1} \) \((Q_{\varepsilon^{-1}}(z)=0)\);

ii) for \( \varepsilon < 0 \), problem (1) is solvable in the class \( K^p(\Gamma) \) if and only if
\[
\int_{\Gamma} \frac{g(t)^k}{X^+(t)} \, dt = 0, \, k = 0,1,\ldots,|\varepsilon|-1;
\] (5)

and under these conditions problem (1) has the unique solution given by (3) with \( Q_{\varepsilon^{-1}}=0 \).

In forthcoming papers we will solve problem (1) in the case of oscillating coefficients \( G \).

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3. Conclusion

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