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Astronomy

On Plane Meridional Orbits for some Forms of Stekkel's Potential

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ABSTRACT. Based on some interesting forms of the Stekkel potential, a problem is posed to investigate orbits with zero kinetic moment with respect to the symmetry axis of the stationary, rotationally symmetric stellar system. Special attention is given to the method of constructing such orbits by taking into account the isolating motion integrals typical only of the well-known Kuzmin's plane model. *© 2010 Bull. Georg. Natl. Acad. Sci.*

Key words: model, orbit, ellipse, invariant, potential, diagram, meridian.

1. Some general aspects of meridional motion. In the field of a gravitational potential of the stellar system, one of interesting classes of plane orbits consists of orbits lying in an arbitrary plane (θ = const) containing the symmetry axis $R = 0$ of the system itself. Here it is assumed that θ is the azimuthal angle in the system of cylindrical coordinates (R,θ,z). As different from another class of orbits lying in the symmetric plane ($z = 0$) of the system, we can call these orbits meridional plane orbits. As is known from the theory of the third quadratic integral of motion [1], the Shtekkel potential in the system of elliptic coordinates ξ_1 and ξ_2 has the form

$$\Phi = \frac{\varphi(\xi_1) - \varphi(\xi_2)}{\xi_1^2 - \xi_2^2}, \tag{1}$$

where $\xi_1 \in [1,\infty[$ and $\xi_2 \in [-1,1]$ are respectively related to the variables R and z by the following transformation formulas

$$R = z_0\sqrt{(\xi_1^2 - 1)(1 - \xi_2^2)}, \quad z = z_0\xi_1\xi_2. \tag{2}$$

It is convenient to introduce one variable $\xi \in [-1,\infty[$ that formally combines the variables ξ_1 and ξ_2 which look like a continuation of each other. It should also be noted that $\varphi(\xi)$ is an arbitrary function of its argument. For the potential form (1) there exist the isolating motion integrals: energy integral E, kinematic moment integral with respect to the symmetry axis K, and third quadratic integral I with respect to velocity components. Here we will consider the case $K \equiv 0$, i.e. orbits with zero kinetic moment.

Differential equations of meridional plane motion can be written as [2]

$$\frac{d\xi_1}{w(\xi_1)} - \frac{d\xi_2}{w(\xi_2)} = 0, \quad \frac{\xi_1^2 d\xi_1}{w(\xi_1)} - \frac{\xi_2^2 d\xi_2}{w(\xi_2)} = \frac{dt}{z_0}. \tag{3}$$

The parameter z_0 defines the position of foci of the elliptic coordinates on the symmetry axis, at which $\xi_1 = 1$, $\xi_2 = \pm 1$ and $R = 0$, $z = \pm z_0$. The function has the form

$$w^2(\xi) = 2[E\xi^2 + 2\varphi(\xi) - I](\xi^2 - 1), \tag{4}$$

where $E < 0$ for finite orbits, and $I \geq 0$.

We obviously have

$$w^2(\xi) \geq 0. \tag{5}$$

This condition defines the boundaries of the domain filled up with an open meridional orbit. By analogy with the quite useful Lindblad diagram from the theory of orbits lying in the symmetry plane of the axially symmetric system (z = 0) or of orbits in spherical systems, we can introduce here the characteristic diagram (I,-E) on which we have the family of characteristic straight lines

$$E\xi^2 + 2\varphi(\xi) - I = 0. \tag{6}$$

Parametric equations of the envelope of this family have the form

$$\left\{\begin{array}{l} E\xi^2 + 2\varphi(\xi) = I \\ \dfrac{\varphi'(\xi)}{\xi} = -E \end{array}\right\}. \tag{7}$$

In the case of Kuzmin's plane model we have

$$\varphi(\xi) = \frac{GM}{z_0} |\,\xi\,| \tag{8}$$

and equation (6) takes the form

$$E\xi^2 + 2\frac{GM}{z_0} |\,\xi\,| - I = 0. \tag{9}$$

After excluding $|\xi|$ from equation (7), we obtain a simple enough relation

$$I(-E) = \left(\frac{GM}{z_0}\right)^2, \tag{10}$$

where G is the gravitational constant and M is the total mass of Kuzmin's plane model.

It is obvious that the domain of admissible motions on the diagram (I, -E) wholly lies in the first quadrant of the diagram which is bounded from above by a branch of the equilateral hyperbola and from the left by the axis I = 0.

2. The method of constructing meridional orbits in the potential field of Kuzmin's plane model. The potential of Kuzmin's plane model can be written as

$$\Phi(R, z) = \frac{GM}{S}, \tag{11}$$

$$s = \sqrt{R^2 + (|\,z\,| + z_0)^2}, \tag{12}$$

i.e. motion of a test particle in the field of this potential is such that the total mass M of the model seems to be concentrated at the most distant focus. When this test particle passes through the symmetry plane z = 0, the gravitation centre changes abruptly by jumping from focus to another one, the particle motion velocity changes gradually, while the acceleration changes in jumplike manner for z = 0. It is obvious that in this case the orbit is a set of pieces of various Kepler ellipses. In the meridional domain ξ = const, parametric equations of each ellipse piece have the form

$$s = \frac{p}{1 + e\cos(u - \omega)} = \frac{a(1 - e^2)}{1 + \cos(u - \omega)}, \tag{13}$$

$$|z| + z_0 = s\sin u. \tag{14}$$

Here a, e, ω are Kepler elements of the respective ellipse piece, and the variable u is the parameter.

In the case of plane meridional orbits there exist the following three motion invariants:

$$c_1 = z_0^2 a^{-1} = z_0^2 \mathrm{p}^{-1}(1-\mathrm{e}^2), \tag{15}$$

$$c_3 = \mathrm{p} - 2ze\sin\omega, \tag{16}$$

related to the motion integrals E and I by

$$E = -\frac{GM}{z_0^2} c_1, \;\; I = GM(c_3 - c_1). \tag{17}$$

It should be kept in mind that here

$$GMc_2 = K^2 = GMp\cos^2 i = 0.$$

After excluding the parameter u from equations (13) and (14), for the considered ellipse piece we obtain the equation

$$2z_0(s - p) + R\sqrt{4z_0^2 - 4c_1 p - (p - c_3)^2} + $$

$$+ (p - c_3)(|\,z\,| + z_0) = 0. \tag{18}$$

If the value z_0 is assumed to be the unit of length, then for z = 0 we can write the equation

$$[(p - c_3)^2 + 4c_1 ps^2 - 4(p + c_3)s + 4[(c_3 - c_1)p + 1] = 0. \tag{19}$$

In the coordinate system (p, s), equation (19) gives a closed algebraic curve of fourth order which can conditionally be called "a link-up oval" because it completely solves the problem of passage from one ellipse piece to another. Such a graphic procedure simplifies the construction of the entire orbit with given c_1 and c_3. Solving equation (19) with respect to the unknown s we obtain

$$s = 2\frac{(p-c_3) \pm \sqrt{(p-c_3) - [p(c_3-c_1)+1][(p-c_3)^2 + 4c_1 p]}}{[(p-c_3)^2 + 4c_1 p]} \quad (20)$$

using which we construct, on the plane (p, s), "a link-up oval" for each pair of invariant motion values.

For all pairs of values of the invariants c_1 and c_3 related by the formula

$$c_3(c_3 - c_1) = z_0^2, \quad (21)$$

we have periodic orbits with equal periods with respect to R and z or ξ_1 and ξ_2. Hence we find that

$$c_3 = 0{,}5(c_1 + \sqrt{c_1^2 + 4z^2}\ ). \quad (22)$$

For $c_2 = 0$, the limit curves in the plane of invariants (c_1, c_3) are defined by the equation

$$[4z_0^{\ 2} - 4c_1 p - (p - c_3)^2]p = 0, \quad (23)$$

the roots of which are

$$p = c_3 - 2c_1 \pm 2\sqrt{z_0^2 - c_1(c_3 - c_1)}, \quad p = 0. \quad (24)$$

Among them we should choose two largest roots. We obviously have

$$c_1(c_3 - c_1) \le z_0^{\ 2} \ \text{ or } \ c_3\, d \le z_0^{\ 2} c_1^{\ -1} + c_1. \quad (25)$$

For $c_1 > z_0$, at least one root of the quadratic equation

$$4z_0^{\ 2} - 4c_1 p - (p - c_3)^2 = 0 \quad (26)$$

is negative and should be replaced by $p = 0$. In order that the other root of equation (26) be non-negative, we should take $c_3\ d \le 2z_0$. Thus for the limit curves we respectively obtain

$$c_3 = z_0^{\ 2} c_1^{\ -1} + c_1, \quad p = z_0^{\ 3} c_1^{\ -1} - c_1, \quad c_1 = z_0 e d \le z_0, \quad (27)$$

$$c_3 = 2z_0, \quad p = 0, \quad c_1 \ge z_0, \quad e = 1. \quad (28)$$

In the general case, where $c_2 = 0$, the limit surfaces in the space of three motion invariants (c_1, c_2, c_3) are defined by the equation

$$4(z_0^{\ 2} - c_1 p)(p - c_2) - (p - c_3)^2 p = 0, \quad (29)$$

which can be written in the standard form

$$p^3 - 2(c_3 - 2c_1)p^2 + [c_3^{\ 2} - 4(c_1 c_2 + z_0^{\ 2})]p + 4z_0^{\ 2} c_2 = 0.$$

It should be noted that one of the roots of this equation is always negative. Solving this equation together with the equation

$$3p^2 - 4(c_3 - 2c_1)p + c_3^{\ 2} - 4(c_1 c_2 + z_0^{\ 2}) = 0 \quad (30)$$

we find that

$$c_2 - 2c_1 = p - 2z_0^{\ 2} c_2 p^{-2}, \quad (31)$$

$$c_3^2 - 2c_1 c_2 = p^2 - 2c_2 p + 4z_0^{\ 2} p^{-2}(p - c_2)^2, \quad (32)$$

$$c_3 - c_2 = (p - c_2)p^{-1}\sqrt{4z_0^2 + p^2}\ . \quad (33)$$

These equations yield the expression

$$c_1 = z_0^{\ 2} c_2 p^{-2} + 0.5(p - c_2)p^{-1}(\sqrt{4z_0^2 + p^2} - p). \quad (34)$$

For the motion invariants c_1, c_2 and c_3 we have the relations

$$p - c_2 = (c_3 - c_2)\frac{p}{\sqrt{4z_0^2 + p^2}}, \quad (35)$$

$$2(c_1 p - z_0^2) = \left(1 + \frac{2z_0^2 + p^2}{p\sqrt{4z_0^2 + p^2}}\right)(c_3 - c_1), \quad (36)$$

from which it follows that the isolines of these invariants are straight lines. Using these formulas, one can easily calculate isolines c_2. In the space (c_1, c_2, c_3), isolines p are also straight lines (on the limit surface).

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