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Mathematics

On the Notion of Generalized Spline for a Sequence of Problem Elements Sets

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ABSTRACT. In the present paper the notion of spline is generalized for the case where not one set of problem elements, but a decreasing sequence of problem elements sets on a linear space is given. The generalized interpolation spline realizes a minimum not only of the metric, but also of the corresponding Minkowski functional. The necessary and sufficient condition for the existence of generalized spline for arbitrary nonadaptive information of cardinality 1 is given. *© 2010 Bull. Georg. Natl. Acad. Sci.*

Key words: *generalized interpolation spline, Problem element set, Frechet space, strong proximality.*

A polynomial spline is defined as a piecewise polynomial function having some smoothness at given knots. Such functions represent solutions of some extremal problems. For Hilbert spaces, splines are defined and studied in the monograph by P. Laurent [1]. For Banach spaces similar problems are studied by R. Holmes [2], D.Ugulava [3] and others. We will use the definition of splines which is given in the monograph by J.Traub, G.Wasilkowski and H.Wozniakowski [4]:

Let F_1 be a linear space, and $F \subset F_1$ be a set of problem elements, which is balanced and convex (absolutely convex). The set F is generated by means of the restriction operator $T: F_1 \to X$, where X is a normed space and $F = \{ f \in F_1; \| Tf \|_X \le 1\}$. F_1 and X are linear spaces over the real field. Let $I: F_1 \to R^m$ be nonadaptive information of cardinality m, having the form $I(f) = [L_1(f), \cdots, L_m(f)]$, where $L_1, \cdots, L_m$ are linear functionals on F_1. Let $y \in I(F_1)$. An element $\sigma = \sigma(y)$ is called a spline interpolating y (briefly, a spline) iff $I(\sigma) = y$ and $\| T\sigma \| = \min\{\| Tz \|;\, z \in F_1, I(z) = y\}$. Thus σ is an element that interpolates the data and has a minimal $T-$norm among all elements interpolating y. A necessary and sufficient condition for the existence of a spline is given in ([4], p. 96). This condition can be written in terms of the Minkowski functional μ_F of the set F as follows.

Theorem 1 [5]. *For nonadaptive information I of cardinality m, the operator T and a spline interpolating $y \in I(F_1)$ exists iff the m-codimensional subspace KerI is proximal in F_1 with respect to the Minkowski functional μ_F of the set F.*

Based on this result, in the present paper this notion is generalized for the case where not one set of problem elements, but a decreasing sequence of problem elements sets is given on the space F_1 (in what follows, as different from [4], the space F_1 will be denoted by E). Let E be a locally convex metrizable space with an increasing sequence of seminorms $\{ || . ||_n \}$ generating the topology. It is well known that there exists an invariant with respect to the translation metric d with absolutely convex balls $K_r = \{x \in E;\, d(x,0) \le r\}$, such that (E,d) is a linear metric space. By $q_r(\cdot)$ we denote the Minkowski functional of a ball K_r, and by $d(x,0) = |\cdot|$ the quasinorm.

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Metrics with convex balls were constructed by Albinus [6] who called them normlike metrics. D.Zarnadze [7] constructed a metric having the following form

$$d(x,y)=\begin{cases} \|x-y\|_1 \ \text{ if } \ \|x-y\|_1\geq 1, \\ 2^{-n+1} \ \text{ if } \ \|x-y\|_n\leq 2^{-n+1} \text{ and } \|x-y\|_{n+1}\geq 2^{-n+1} \ (n\in N), \\ \|x-y\|_{n+1} \ \text{ if } \ 2^{-n}\leq \|x-y\|_{n+1}\leq 2^{-n+1} \ (n\in N), \\ 0 \ \text{ if } \ x-y=0. \end{cases} \tag{1}$$

Minkowski functionals $q_r(\cdot)$ for the ball K_r of the metric (1) are dependent on the initial seminorms through the following simple equality [7]

$$q_r(\cdot)=r^{-1}\|\cdot\|_n, \text{ where } r\in I_n=\begin{cases} [1,\infty[\ \text{ if } \ n=1, \\ [2^{-n+1},2^{-n+2}[\ \text{ if } \ n\geq 2. \end{cases} \tag{2}$$

Thus $K_r=rV_n$, where $V_n=\{x\in E;\|x\|_n\leq 1\}$, if $r\in I_n$. For any $n\in N$ we consider V_n as a set of problem elements.

Let $I:E\to R^n$ be nonadaptive information of cardinality m, $y\in I(E)$, $I(f)=y$ for some $f\in E$ and $d(f,KerI)=r$. Then an element $\sigma=\sigma(y)=f-h^*$ is said to be a generalized spline interpolating y (briefly, a generalized spline) if $I(\sigma)=y$,

$$d(f,KerI)=d(f,h^*)=r=d(\sigma,0)=|\sigma| \tag{3}$$

and

$$\inf\{q_r(f-h)\colon h\in KerI\}=q_r(f-h^*)=q_r(\sigma). \tag{4}$$

The generalized interpolation spline realizes a minimum not only of the metric, but also of the corresponding Minkowski functional. In other words, the generalized interpolating spline exists iff the subspace $KerI$ is strongly proximal in the metric space (E,d). This notion was introduced by us in [8]. For the normlike metrics the conditions (3) and (4) are equivalent and $q_r(\sigma)=1$. Therefore, in that case the notion of strong proximality coincides with that of ordinary proximality. For the metric (1) this notion takes the following form: $\sigma=\sigma(y)=f-h^*$ is said to be a generalized spline interpolating y if $I(\sigma)=y$,

$$d(f,KerI)=d(f,h^*)=r=d(\sigma,0)=|\sigma| \qquad \text{if} \qquad r\in \operatorname{int} I_n, \tag{5}$$

and

$$\inf\{\|f-h\|_n : h\in KerI\}=\|f-h^*\|_n=\|\sigma\|_n\leq r \quad \text{if} \quad r=2^{-n+1} \ (n\in N). \tag{6}$$

From the property (2) of the metric (1) we have that for $r\in \operatorname{int} I_n$ the fulfillment of the condition (5) is sufficient for σ to be a generalized interpolating spline. In the case $r=2^{-n+1}$ $(n\in N)$, (5) follows from (6), but, in general, the best approximating element with respect to the metric may not have an analogous property with respect both to $q_r(\cdot)$ and to $\|\cdot\|_n$ simultaneously [9]. If $V_n=\{x\in E:\|x\|_n\leq 1\}$ and $V_1=\dots=V_n=\dots=F$, then $K_r=rF$, $|\cdot|=\mu_F(\cdot)$ and the notion of generalized interpolating splines coincides with the classical une.

The problem of the existence of generalized interpolating splines naturally arises for arbitrary nonadaptive information of cardinality m and $y\in I(E)$. For $m=1$ this problem is equivalent to the strong proximality for each closed hypersubspace in an arbitrary Frechet space. In the case of a Banach space the answer to this question is given by the well-known James theorem, according to which a Banach space is reflexive iff its every closed hypersubspace is proximal, i.e. if there exists interpolating spline for every nonadaptive information of cardinality 1 and arbitrary $y\in I(E)$. In the case of a Banach space the existence of interpolating splines for arbitrary nonadaptive information of cardinality 1 includes the existence of interpolating splines for arbitrary nonadaptive information of cardinality m and arbitrary $y\in I(E)$. Indeed, in that case the space will be reflexive and therefore every closed subspace will be proximal.

The problem of (strong) proximality of all closed hypersubspaces with respect to the normlike metrics in the Frechet spaces was considered by many mathematicians. D. Zarnadze [9] found and described the exact class of Frechet spaces, in which every closed hypersubspace is (strongly) proximal with respect to the normlike metrics. This is the class of reflexive strictly regular Frechet spaces, which coincides with the class of reflexive quojections.

A Frechet space E is said to be a quojection if for any continuous seminorm $p(\cdot)$ defined on E the factor space $E/Kerp(\cdot)$ is normed.

Theorem 2. *Let E be a Frechet space, whose topology is generated by an increasing sequence of seminorms $\{\|\cdot\|_n\}$ and by the above mentioned normlike metric d. Then the following assumptions are equivalent:*

a) every closed hypersubspace is (strongly) proximal with respect to the normlike metrics;

b) there exists a generalized interpolating spline for arbitrary nonadaptive information of cardinality 1 and arbitrary $y \in I(E)$;

c) the Frechet space E is a reflexive quojection.

Proof. a) $\Leftrightarrow$ b) follows from the above-mentioned reasoning. a) $\Leftrightarrow$ c) is proved in [10] (Theorem 1).

A similar result is valid for the metric (1). Namely, the following statement is true.

Theorem 3. *Let E be a Frechet space, whose topology is generated by an increasing sequence of seminorms $\{\|\cdot\|_n\}$ and by the metric (1). Consider the following assumptions:*

a) every closed hypersubspace is strongly proximal with respect to the metric (1);

b) a generalized interpolating spline exists for arbitrary nonadaptive information of cardinality 1 and arbitrary $y \in I(E)$;

c) every closed hypersubspace is proximal with respect to the metric (1);

d) the Frechet space E is a reflexive quojection.

Proof. a) $\Leftrightarrow$ b) follows from the definition of a generalized interpolating spline. c) $\Leftrightarrow$ d) is proved in [9] (Theorem3). a) $\Rightarrow$ c) is trivial.

In [9], some classes of reflexive Frechet spaces are indicated, in which there exist nonproximinal closed hypersubspaces. This is equivalent to the existence of nonadaptive information of cardinality 1, for which the generalized interpolating spline does not exist.

Unlike Banach spaces, in Frechet spaces the proximality of all closed hypersubspaces does not imply the proximality of all subspaces. Therefore from the existence of interpolating splines for arbitrary nonadaptive information of cardinality 1 does not follow the existence of interpolating splines for arbitrary nonadaptive information of cardinality $m > 1$ and arbitrary $y \in I(E)$.

Theorem 4. *Let E be a Frechet space with an increasing sequence of Hilbertian seminorms $\{\|\cdot\|_n\}$, $V_n = \{x \in E : \|x\|_n \le 1\}$ and with the metric (1). Let $K_n : E \to E / Ker\|\cdot\|_n$ be a canonical mapping, $E_n = (E/Ker\|\cdot\|_n, \|\cdot\|_n^{\wedge})^{\sim}$, where $\|\cdot\|_n^{\wedge}$ is the associated norm and $X^{\sim}$ denotes completion of X, and H be an infinitedimensional closed subspace of E. Then the following assertions are valid:*

a) if $K_n(H)$ is closed in the Hilbert space E_n for any $n \in N$, then H is strongly proximal in E with respect to the metric (1)

b) if for nonadaptive information I of cardinality m the subspace $K_n(KerI)$ is closed in the Hilbert space E_n, $n \in N$, then for any $y \in I(E)$ there exists a generalized spline interpolating y.

Proof. a) Let $f \in E \setminus H$ and $d(f,H) = r \in I_n$ for some $n \in N$. By the property (2) of the metric (1) we have $\inf\{\|f-h\|_n;\ h \in H\} = \lambda \le r$ (more precisely, if $r \ne 2^{-n+1}$ $(n \in N)$, then $\lambda = r$) [7]. Therefore $\inf\{\|K_n f - K_n h\|_n^{\wedge};$ $h \in H\} = \lambda \ge 0$. For $\lambda > 0$, since $K_n(H)$ is closed in the Hilbert space E_n, $K_n(H)$ is proximal in E_n and there exists an element h^* such that $\inf\{\|K_n f - K_n h\|_n^{\wedge};\ h \in H\} = \|K_n f - K_n h^*\|_n^{\wedge} = \lambda$. This means that h^* is the best approximation element for f with respect to $q_r(\cdot)$ and therefore a strongly proximal element for f. If $r = 2^{-n+1}$ and $\lambda \ge 0$, then $f - h_0 \in V_n$ for some $h_0 \in H$. Indeed, in that case there exists a minimizing sequence $h_k \in H$ such that $Lim_{k\to\infty}\|K_n f - K_n h_k\|_n^{\wedge} = \lambda$. Then this sequence $\{K_n h_k\}$ will be bounded in E_n. Its some subsequence will be weakly convergent to an element $x_0 \in E_n$. The set $K_n(H)$ is closed and therefore $x_0 = K_n h_0$. Thus $f - h_0 \in V_n$. Let us suppose that $f - h_0 \in 2V_{n+1}$. Then we have $d(f,h_0) \le 2^{-n+1}\|f-h_0\|_{n+1} < 2^{-n+1}$, which is impossible. Therefore $\|f-h_0\|_n \le 2^{-n+1}$ and $\|f-h_0\|_{n+1} \ge 2^{-n+1}$. This means that $d(f,h_0) = r = 2^{-n+1}$. It is also clear that $Lim_{k\to\infty}\|K_n f - K_n h_k\|_n^{\wedge} = \|K_n f - K_n h_0\|_n = \ = \lambda = \|f-h_0\|_n$.

b) follows from a) if we replace H by $KerI$.

For *m*>1 and a general Frechet space we do not have an exact characterization for the existence of generalized interpolating splines in the case of arbitrary nonadaptive information of cardinality *m* and $y \in I(E)$. In [9] a necessary and sufficient condition is given for an arbitrary closed subspace of a Frechet space to be proximal. It is interesting to find such classes of Frechet spaces in which every closed subspace is strongly proximal. For example, such is the space $B \times \omega$, where B is a reflexive Banach space and ω is a nuclear Frechet space of all sequences. In the case of Frechet spaces, from the existence of generalized interpolating splines for nonadaptive information of cardinality 1 and arbitrary $y \in I(E)$ we do not obtain an analogous result for nonadaptive information of cardinality *m*>1.

Theorem 5. *Let E be a Frechet space with an increasing sequence of Hilbertian seminorms $\{\|\cdot\|_n\}$, $V_n = \{x \in E : \|x\|_n \le 1\}$ and with the metric (1). Let $K_n : E \to E/Ker\|\cdot\|_n$ be a canonical mapping, $E_n = (E/Ker\|\cdot\|_n, \|\cdot\|_n^{\wedge})^{\sim}$ and I be nonadaptive information of cardinality $m \ge 1$. If the subspace $KerI$ possesses an orthogonal complement in E and $K_n(KerI)$ is closed in a Hilbert space E_n, $y \in I(V_1)$, $f_0 \in I^{-1}(y) \cap V_1$, $d(f_0, KerI) = r \in I_{n_0}$, then there exists a unique generalized interpolating y spline $\sigma = \sigma(y)$, which is a symmetric center for all sets $I^{-1}(y) \cap V_n$ $(n \le n_0)$.*

Proof. For any $y \in I(E)$ and information I we take f such that $I(f) = y$. Since the subspace $KerI$ possesses an orthogonal complement in E, there exists a unique representation $f = h^* + \sigma$ and $(h^*, \sigma)_n = 0$ for any $n \in N$, where $h^* \in KerI$ and $\sigma \in KerI^{\perp}$. This means that $< K_n h^*, K_n \sigma >_n = 0$ for any $n \in N$, where $< \cdot, \cdot >$ denotes an inner product in the space E_n and σ is the unique best approximation element for f in $KerI$ with respect to the seminorm $\|\cdot\|_n$ for any $n \in N$. $K_n(h^*)$ is the best approximation element in $K_n(KerI)$. By Theorem 4 this means, that $\sigma = f - h^*$ is a generalized spline interpolating y.

Let us assume that $y \in I(V_1)$, and $f \in I^{-1}(y) \cap V_1$. Then $f = h + \sigma$, where σ is the above-constructed unique generalized spline interpolating y, $h \in KerI$ and $d(f, h) = d(\sigma, 0) = r \in I_{n_0}$. By the above-mentioned property of the metric (1) we have $q_r(\sigma) = r^{-1}\|\sigma\|_{n_0} \le 1$, and therefore $\inf\{\|f - g\|_{n_0};\ g \in KerI\} = \|f - h\|_{n_0} = \|\sigma\|_{n_0} \le r$. Thus $\sigma = \sigma(y)$ is the center of all sets $V_n^{'} = I^{-1}(y) \cap V_n$, $n \le n_0$, i.e. $f \in V_n^{'}$ implies that $2\sigma(y) - f \in V_n^{'}$. Indeed, this follows from the fact that $I(2\sigma - f) = y$ and for $h = \sigma - f \in KerI$ we have $(\|K_n(2\sigma - f)\|_n^{\wedge})^2 = (\|K_n(\sigma) + K_n(h)\|_n^{\wedge})^2 = (\|K_n(\sigma)\|_n^{\wedge})^2 + (\|K_n(h)\|_n^{\wedge})^2 = (\|K_n(f)\|_n^{\wedge})^2 = (\|f\|_n)^2$, i.e. $2\sigma - f \in V_n'$. Therefore the set V_n' is symmetric with respect to σ for all $n \le n_0$.

The question arises whether there are cases in which a classical spline does not exist but a generalized spline does. Below we give the answer to this question.

Let E be a Frechet space with an increasing sequence of Hilbertian norms $\{\|\cdot\|_n\}, V_n = \{x \in E : \|x\|_n \le 1\}$, i.e. $\|\cdot\|_n$ be the Minkowski functional of V_n. Let E_n be the normed space $E_n = (E, \|\cdot\|_n)$. If instead of F we consider the set V_n for each $n \in N$, then the identical maps $K_n : E \to E_n$ will be analogs of the operator $T : F \to X$. Let us consider nonadaptive information I. The spline corresponding to this information and to the operator K_n, if it exists, is denoted by σ_n. It is well known that the dual space $E^{'}$ of E is represented as $E^{'} = \bigcup_{n \in N} E^{'}_{V_n^0}$, where $E^{'}_{V_n^0}$ is the Hilbert space which is spanned on the polar V_n^0. It is well known that this space $E^{'}_{V_n^0}$ is isomorphic to the space $E_n^{'}$. Therefore, if we have information of cardinality 1 generated by a continuous function $L_1(f)$, then there exists $n_0 \in N$ such that $L_1(f)$ belongs to any $E_n^{'}$, $n \ge n_0$. Hence $KerI$ will be closed with respect to the norms $\|\cdot\|_n$ when $n \ge n_0$. For $n < n_0$ the subspace $KerI$ will be not proximal in $(E, \|\cdot\|_n)$. This follows from the fact that if $KerI$ is not closed in $(E, \|\cdot\|_n)$, $n < n_0$, then it is dense in E. Thus, in the latter space the best approximate element does not exist in $KerI$ for some $x \in E$. So, a classical spline does not exist for $n < n_0$. The spline σ_n exists for any number $n \ge n_0$, while $KerI$ is closed and therefore proximal in $(E, \|\cdot\|_n)$, $n \ge n_0$.

მათემატიკა

განზოგადებული სპლაინის ცნების შესახებ პრობლემის ელემენტების სიმრავლეთა მიმდევრობისათვის

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ნაშრომში სპლაინის ცნება განზოგადებულია იმ შემთხვევისათვის, როდესაც წრფივ სივრცეზე მოცემულია პრობლემის ელემენტთა არა ერთი სიმრავლე, არამედ პრობლემის ელემენტების სიმრავლეთა მიმდევრობა. განზოგადებული საინტერპოლაციო სპლაინი მინიმუმს ანიჭებს არა მხოლოდ მეტრიკას, არამედ შესაბამის მინკოვსკის ფუნქციონალსაც. მიღებულია განზოგადებული სპლაინის არსებობის აუცილებელი და საკმარისი პირობა 1 კარდინალობის ნებისმიერი არაადაპტური ინფორმაციისათვის.

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