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Mathematics

Greedy Algorithm Fails in Compact Vector Summation

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ABSTRACT. We show that in any two-dimensional normed space there exists a collection of vectors $x_1, x_2, \ldots, x_n$, $n \geq 1$, such that the greedy algorithm for estimation of $\min_{\pi} \max_{1\leq k\leq n} \left\| \sum_{i=1}^{k} x_{\pi(i)} \right\|$ fails to be optimal.

Key words: *normed space, greedy algorithm, optimal algorithm.*

Let X be a linear normed space and let $x_1, x_2, \ldots, x_n$, $n \geq 1$, be a collection of vectors of X. Given a permutation $\pi : \{1,\ldots,n\} \to \{1,\ldots,n\}$ consider the number $\varphi(\pi) = \max_{1\leq k\leq n} \left\| \sum_{i=1}^{k} x_{\pi(i)} \right\|$. In many applications, for example in a series of problems of scheduling theory (see e.g. [1-3]), it is of importance to find a permutation $\pi_{optimal}$ for which $\varphi(\pi)$ attains its minimum. We call such a permutation optimal. In scheduling theory the arrangement of summands corresponding to the $\pi_{optimal}$ is called the compact vector summation. Estimation of $\varphi(\pi_{optimal})$ is frequently called the problem of (dynamical) compact vector summation (CVS).

For their simplicity and constructiveness the so-called greedy algorithms in general seem to be effective. In our case the greedy algorithm (greedy permutation) to approach $\varphi(\pi_{optimal})$ goes as follows: on step one we take from the collection $x_1, x_2, \ldots, x_n$ an element x_{n_1} having the minimum norm, on step two we take x_{n_2} such that $\| x_{n_1} + x_{n_2} \|$ is minimal, etc. Note that for the computational purposes this algorithm is of importance for it runs in polynomial time.

Naturally the question arises as to whether the greedy algorithm is of the same order as the optimal one, i.e. there exists a constant C, dependent only on the space X, such that $\varphi(\pi_{greedy}) \leq C\varphi(\pi_{optimal})$. It can be shown that in the one-dimensional case the constant C equals 2. As to the multi-dimensional case, Jakub Wojtaszczyk (oral communication) has constructed the following systems of vectors in l_∞^2 for which $\frac{\varphi(\pi_{greedy})}{\varphi(\pi_{optimal})}$ is not bounded.

Example. Consider the collection of n blocks each consisting of the following three vectors of l_∞^2 : (1,1), (2,-3) and (-3,2). Obviously, for this system the greedy algorithm starts with n vectors (1,1) in a row. This implies that $\varphi(\pi_{greedy}) \geq n$. Whereas $\varphi(\pi_{optimal}) = 3$ (we note that the sum of the three vectors vanishes).

The main question of this note is as to whether the same sort of example can be constructed for an arbitrary 2-dimensional normed space.

Theorem. *For any two-dimensional normed space X (real or complex) and any prescribed number M there exists a collection* $x_1, x_2, \ldots, x_n$, $n \geq 1$, *of vectors of X such that* $\dfrac{\varphi(\pi_{greedy})}{\varphi(\pi_{optimal})} > M$.

Proof. Let H be a two-dimensional linear inner product space with a norm $\|\cdot\|_H$ and let $T: X \to H$ be a non-degenerate linear operator with $\|T\| = 1$. Denote by $a \in X$ a vector such that $\|a\|_X = \|Ta\|_H = 1$ and introduce $h_1 = Ta$, a vector $h_2 \in X$ orthogonal to h_1 with $\| h_2 \|_H = 2\sqrt{n}$; $b = T^{-1}h_2$; $x_1 = -\dfrac{1}{\sqrt{n}}a$; $x_2 = \dfrac{1}{\sqrt{n}}(a+b)$; $x_3 = \dfrac{1}{\sqrt{n}}(a-b)$. The collection in question consists of n following 4-element groups: x_1, x_2, x_3, x_1. Obviously, each of the groups sums up to zero. The greedy algorithm chooses x_1 as the first vector, because $\| x_1\|_X = \dfrac{1}{\sqrt{n}}$, while $\| x_2\|_X > 2$ and $\| x_3\|_X > 2$. Indeed,

$$\| x_2 \|_X = \frac{1}{\sqrt{n}} \| a+b \|_X > \frac{1}{\sqrt{n}} \| h_1 + h_2 \|_H = \frac{1}{\sqrt{n}} \sqrt{\| h_1 \|_H^2 + \| h_2 \|_H^2} = \frac{1}{\sqrt{n}} \sqrt{1+ 4n} > 2 .$$

The following calculation shows that on each of the subsequent n steps the greedy algorithm also selects x_1 . We have for any $k = 1,2,\ldots,n$:

$$\| (k-1) x_1 + x_2\|_X = \frac{1}{\sqrt{n}} \| (-k+2) a + b\|_X \geq \frac{1}{\sqrt{n}} \| (-k+2) h_1 + h_2\|_H > \frac{k}{\sqrt{n}} = \| k x_1\|_X ,$$

and analogously $\| (k-1)x_1 + x_3\|_X > \|kx_1\|_X$. This suggests that for the given collection we have $\varphi(\pi_{greedy}) \geq \sqrt{n}$, whereas for the arrangement $\overbrace{x_1, x_2, x_3, x_1, x_1, x_2, x_3, x_1, \ldots, x_1, x_2, x_3, x_1}^{4n}$ we get $\varphi(\pi) \leq 2\|T^{-1}\|$. Therefore, $\varphi(\pi_{optimal}) \leq 2\|T^{-1}\|$ and taking $n > 4M^2 \|T^{-1}\|^2$ we get $\dfrac{\varphi(\pi_{greedy})}{\varphi(\pi_{optimal})} > M$.

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მათემატიკა

"ხარბი" ალგორითმი არ არის საკმარისი ვექტორთა კომპაქტური ჯამებადობისათვის

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ნაშრომში ნაჩვენებია, რომ ნებისმიერ ორგანზომილებიან ბანახის სივრცეში არსებობს ვექტორთა ერთობლიობა $x_1, x_2, \ldots, x_n,\ \ n \geq 1,$ **რომლისთვისაც** $\min\limits_{\pi} \max\limits_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} x_{\pi(i)} \right\|$ **გამოსახულების შეფასება "ხარბი" ალგორითმის გამოყენებით არ არის ოპტიმალური.**

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