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Mathematics

Remarks on Spherical Linkages

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ABSTRACT. We present several results on moduli spaces of spherical geodesic linkages. It is established that the signed area function is generically a Morse function on the moduli space of a moderate spherical linkage and its critical points are given by cyclic configurations of the linkage. Next, we present a number of results on cyclic and tangential configurations of open spherical linkages. In particular, we give an explicit formula for the spherical area of a cyclic spherical quadrilateral in terms of the lengths of its sides. Moreover, we prove that the end-point map of an open moderate spherical linkage is a stable mapping from the moduli space to the ambient sphere. *© 2010 Bull. Georg. Natl. Acad. Sci.*

Key words: *spherical polygonal linkage, moduli space, Morse function, cyclic configuration.*

1. We begin by introducing the class of spherical linkages which will be considered in this paper. Let $S=S^2$ be a unit sphere in three-dimensional Euclidean space $\boldsymbol{R}^2$ and d be the distance function defined by the induced Riemannian metric on S [1]. Given a natural number k, an open ***spherical linkage*** (or ***spherical geodesic k-chain***) L is defined by a k-tuple of nonnegative real numbers $l_i < \pi$ (called *sidelengths* of L) [2]. The ambient unit sphere S will be fixed throughout the paper and one thinks of such a linkage as a chain of geodesic arcs on S. For brevity and visuality, we will speak of an ***S-arcade linkage*** or simply ***S-arcade***. An S-arcade is called *moderate* if the sum of all sidelengths does not exceed π.

We will consider two versions of the concept of *moduli space of spherical arcade* which are relevant for our purposes. To this end it is convenient to distinguish between *open arcades* (or *spherical mechanical arms*) and *closed arcades* (or *spherical polygonal linkages* [1]). It is handy to denote an open arcade by M and a closed arcade by P. When the discussion is applicable to both classes we will say S-arcade and denote it by P.

The configuration space $C_S(M)$ of an open arcade M is defined as the totality of all k-tuples of points $v_i \in S$ such that $d(v_i,v_{i+1})=l_i$, $i=1,...,k$. In the case of a closed arcade P we additionally require that $v_{i+1}=v_1$. Each such collection of points is called a *configuration* of arcade. Factoring this set over the natural diagonal action of $SO(3)$ one obtains the ***moduli space*** $M_S(L)$ [2]. Thus one can think of configuration as a chain of geodesic arcs of prescribed lengths. As in the planar case, moduli spaces are endowed with natural topologies induced by distance d making them into compact topological spaces [2].

Applying a homothety with center at the origin and coefficient $\boldsymbol{R}$ one obtains a spherical linkage $\boldsymbol{R}*L$ with sidelengths $\boldsymbol{R}l_i$ on sphere $S_{\boldsymbol{R}}$ of radius $\boldsymbol{R}$ and can define its moduli space in $S_{\boldsymbol{R}}$. It is obvious that the moduli spaces of L and $\boldsymbol{R}*L$ are homeomorphic. If we multiply all sidelengths by the same number $r<1$, the corresponding linkage $r*L$ also has the same moduli space in S as L itself. Moduli spaces of spherical linkages seem to remain poorly explored. For this reason we begin with a few general remarks on those moduli spaces following the general approach suggested in [3] which was already used in our previous note [4].

2. Clearly, the moduli space of a moderate open k-arcade is diffeomorphic to (k-1)-torus T^{k-1}. Next, by complete analogy with the planar case, the moduli space of a linkage L as above can be identified with the subset of configurations such that v_1=(1,0,0), v_2=($\cos l_1$,$\sin l_1$,0). Assuming that this is always the case it is easy to see that, for a closed k-chain P, $M_S(P)$ can be represented as a level set of an obvious smooth mapping from $(S^2)^k$ to $\boldsymbol{R}^k$, which is called the *linkage mapping* (cf. [2]). By a standard application of Ehresmann fibration theorem we conclude that, for generic values of l_i, the moduli space $M_S(P)$ has a natural structure of compact orientable manifold of dimension k – 3. It is known that the moduli space is smooth if and only if the closed arcade is *nondegenerate*, i.e. it does not have a configuration all vertices of which lie on the same great circle of S. It is obvious that a closed arcade is degenerate if and only if there exists a k-tuple of "signs" s_i=±1 such that $\Sigma s_i l_i$=0.

In the sequel we only deal with moderate arcades, which guarantees that all geodesic segments appearing in the sequel are well-defined. Obviously, this condition is preserved by homotheties. Notice that all configurations of a moderate linkage in S with the two vertices fixed as above, belong to the same hemisphere of S. Thus the stereographic projection Π of S on the tangent plane at point v_1=(1,0,0) defines a one-to-one mapping on $M_S(L)$. Since for big R the distortion of Π is small compared with sidelengths, it appears possible to indentify the moduli space of moderate spherical arcades with the moduli space of an appropriate planar linkage (see [4]). In this sense the topology of moduli spaces of moderate arcades is the same as for planar linkages and one can compute a lot of topological invariants in terms of sidelengths.

If $M(L)$ is smooth one can investigate its topology by considering an appropriate differentiable function on $M(L)$ and studying its critical points. One of such functions is the signed spherical area A considered in [5] in the case of spherical quadrilaterals. The first main topic of this note is to investigate configurations which are critical points of A.

3. By analogy with the case of planar polygons, under a *cyclic spherical polygon* we understand a polygon which can be inscribed in a circle lying on S, i.e., there exists a point in S (circumcenter) equidistant from all vertices of the polygon (see, e.g., [1]). Analogously, a spherical polygon P is called *tangential* if there exists a circle on S which is tangent to all sides of P. This is naturally applicable to configurations of arcades and so we can speak of cyclic and tangential configurations. A tangential configuration of an open chain will be called *strictly tangential* if the geodesic segment connecting the first and last vertices is tangent to the same circle. Notice that this geodesic segment, as well as all other geodesic segments we will deal with, are well-defined due to the condition that the arcade is moderate. We can now formulate the first two main results which generalize similar results for planar linkages established in [5], [6].

Theorem 1. *Let L be a moderate S-arcade with smooth moduli space $M_S(L)$. Then all critical points of the signed spherical area A considered as a function on $M_S(L)$ are given by cyclic configurations of L.*

We emphasize that this result holds for arbitrary open arcades and nondegenerate closed arcades. In the case of an open chain one can indicate an additional geometric condition which should be satisfied by a cyclic configuration which is a critical point of A (cf. [6]). The scheme of proof is the same as in the planar case. We describe the main steps in the case of closed chain. First, we prove the result for quadrilateral linkages by direct verification using Lagrange multipliers and standard formulas of spherical trigonometry. Next, it is shown that the geometric "four-hinge method" of Steiner [1] is applicable in the spherical case, which enables one to derive the general result from the quadrilateral case. The case of an open linkage is more complicated and requires developing a proper modification of the argument used in [5].

Theorem 2. *For generic sidelengths, A is a Morse function on $M_S(L)$.*

This can be proven using the standard paradigm of parametric transversality theorem (cf. [5]). A natural problem then is to find a way of computing the Morse index of A at a cyclic configuration. For spherical quadrilaterals this was done in [2], which in virtue of [6] solves the case of an open spherical 3-chain, too. It is also possible to give a complete solution for spherical pentagons and 4-chains in terms of the sidelengths and combinatorics of the cyclic configuration.

A relevant combinatorial invariant is the number $N(V)$ of intersection points of the sides in a given configuration V of L. For a pentagon, the moduli space is two-dimensional and it is easy to see that the values of number $N(V)$ can be 0, 1, 2, 5. For N=0 we have a global extremum (maximum or minimum depending on the orientation) and for N=5 a local extremum, while cyclic configurations with N=1, 2 are saddle points (Morse index is equal to 1). Thus for pentagons the Morse indices can be computed similarly to the planar case investigated earlier by G.Bibileishvili and E.Elerdashvili. However, we were unable to get a complete answer for hexagons either in spherical, or in planar case, more so for k-chains with arbitrary $k > 5$.

4. We present now several remarks on cyclic and tangential configurations of spherical arcades. They generalize similar statements for cyclic configurations of planar mechanical arms presented in [5] and can be proved by an evident modification of the argument given in [5]. Notice that a notion of convexity is naturally defined for each subset of S with diameter not exceeding π, in particular, for each configuration of a moderate arcade.

Proposition 1. The set of cyclic configurations of an open moderate arcade is one-dimensional and always contains an arc of convex configurations. A closed moderate arcade always has a unique convex cyclic configuration.

Proposition 2. The set of tangential configurations of an open moderate arcade is one-dimensional and always contains an arc of convex configuration. If arcade is regular it always has a strictly tangential convex configuration.

For a closed 4-arcade, one can obtain a number of results on the geometry of its convex cyclic configuration in the spirit of [9]. Let $P(a',b',c',d')$ be a closed moderate arcade on S and Q its convex cyclic configuration (it is unique by Proposition 1). The radius R of the circumscribed circle of Q in S and its area A can be computed in terms of sidelengths. For positive numbers q, r, s with $2p = q + r + s < \pi$, let $A(q,r,s)$ denote the (nonoriented) area of a spherical triangle with the sides q, r, s. Recall that number $A(q,r,s)$ can be computed by L'Huillier formula ([1], p.73):

$$A(q,r,s) = 4arctg\sqrt{tg\frac{p}{2}tg\frac{p-q}{2}tg\frac{p-r}{2}tg\frac{p-s}{2}}.$$

Proposition 3. For a cyclic quadrilateral $Q(a',b',c',d')$ on S, its circumradius R and spherical area A are given by formulas

$$R = \arcsin\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)}}, \tag{1}$$

$$A = A(R,R,a)+A(R,R,b)+A(R,R,c)+A(R,R,d), \tag{2}$$

where $a = 2\sin\frac{a'}{2}$, $b = 2\sin\frac{b'}{2}$, $c = 2\sin\frac{c'}{2}$, $d = 2\sin\frac{d'}{2}$.

Formula (2) can be considered as a spherical analog of the classical Brahmagupta formula for the area of a cyclic quadrilateral. Comparison of (2) with a more simple-minded direct generalization of the Brahmagupta formula presented in [2] shows that, contrary to the claim in [2], the direct generalization is not reasonable. The proof of the proposition goes as follows.

First, we find the circumradius R. To this end, let C be the circumcenter of Q and OC the corresponding radius of S. Since all vertices of Q lie on a circle, they lie in a certain plane W orthogonal to OC. Let $D = W \cap OC$ be the circumcenter of a planar cyclic quadrilateral Q' formed by the vertices of Q. The sides of Q' are chords of the sides of Q and they can be found by elementary geometry: $a = 2\sin\frac{a'}{2}$, $b = 2\sin\frac{b'}{2}$, etc. By [9] the circumradius r of cyclic quadrilateral Q' can be found by the formula

$$r = \sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)}}.$$

The circumradii of Q' and Q are related by formulas $r = \sin R$, $R = \arcsin r$, which immediately gives formula (1). Since Q is convex, its area is equal to the sum of areas of four spherical triangles with the sides (R,R,a), (R,R,b), (R,R,c), (R,R,d) and formula (2) is also immediate.

We emphasize that formula (2) is quite explicit since each of the summands can be computed by L'Huillier formula. The simple considerations used in the proof are applicable to cyclic arcades with an arbitrary number of sides, which gives a similar explicit formula for the area of an arbitrary cyclic polygon. These formulas enable one to use computer to calculate the Hessian of area and calculate Morse indices along the lines of [3]. Using methods of [3] one can also obtain an explicit formula for the number of different cyclic configurations of a given spherical linkage.

5. In conclusion we present a few remarks concerned with the ***end-point map*** E_M of an open $(k+1)$-arcade M which is defined on the moduli space of M in a natural way. Namely, we fix the position of the first side and identify moduli space $M_S(M)$ with the k-torus T^k consisting of collections α of k angles α_i between the consecutive sides of M. For each such á, one defines $E_M(\alpha)$ to be the point of S obtained as the position of the last vertex. In this way we get a well-defined map $E=E_M$: $T^k \to S$. It is easy to show that E is smooth (infinite differentiable). Hence we may consider its differential $DE(\alpha)$ at a given point $\alpha \in T^k$ which can be interpreted as a linear mapping from $\mathbf{R}^k$ to the two-dimensional tangent plane T_qS at the point $q = E(\alpha)$. Let Ker$DE(\alpha)$ denote its kernel which is a $(k$-2)-dimensional subspace. In a local chart near point q the tangent planes to S can be identified with $\mathbf{R}^2$. Then differential $DE(\alpha)$ is represented by a $(2\times k)$-matrix and its kernel is the kernel of this matrix. We endow T^k with a canonical flat Riemannian metric induced from its universal covering space $\mathbf{R}^k$ and denote by $H(\alpha)$ the orthogonal complement to Ker$DE(\alpha)$. In this way we obtain a two-dimensional distribution $H(E)$ on T^k called the horizontal distribution with respect to E and can consider horizontal lifts of paths in the target [8].

Theorem 3. *For a moderate open spherical k-arm M, the end-point map E_M is stable and its singular set is a union of several concentric circles of fold points.*

Proof. One can use local coordinates and consider the tangent map TE interpreted as a linear map from the universal cover of T^k into a plane representing T_qS. Without loss of generality we can put $q = (1,0,0)$ and identify T_qS with the vertical plane with coordinates (y, z). Then TE can be represented by a $(2\times k)$-matrix whose elements are partial derivatives with respect to angles α_i. Each partial derivative can be computed using the known formulas for infinitesimal variations of the elements of a spherical triangle [1] and vanishing of its (2×2)-minors is easily seen to be equivalent to the fact that each α_i is equal either to 0 or to π. It is now obvious that the critical set consists of several circles obtained by choosing various combinations of these values of angles α_i. From the expressions for the elements of TE it follows that they can never vanish simultaneously, in other words, the rankTE=1 at each critical point. This means that each critical point is a fold point [7] and by the classical result of H. Whitney the map E_M is stable.

We believe that this result is of interest since it is in the spirit of some general constructions from [8] and [9]. In particular, one can apply the general argument from [8] (p. 88) to construct horizontal lifts of paths in the image of E_M and describe the leaves of arising foliation as orbits of an appropriate Lie group. As was shown in [9] in the case of a planar 3-arm, one can also calculate the holonomy Lie algebra at each point of moduli space and derive interesting conclusions about the controllability of configurations. In the case of a closed arcade one can consider similar topics for the *gravicenter mapping* of the moduli space into S defined by sending each configuration to its center of mass. These and other natural developments in the spirit of [8] and [9] are delayed for further publications. Another interesting perspective is to extend the setting of this paper to geodesic linkages in homogeneous spaces of Lie groups.

მათემატიკა

სფერული სახსრული ჯაჭვების შესახებ

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(წარმოდგენილია აკადემიკოს რ. გამყრელიძის მიერ)

ნაშრომში განხილულია სფერული გეოდეზიური სახსრული ჯაჭვის მოდულების სივრცესთან დაკავშირებული ამოცანები. დადგენილია, რომ ორიენტირებული ფართობის ფუნქცია წარმოადგენს მორსის ფუნქციას მოდულების სივრცეზე და მისი კრიტიკული წერტილებია წრეწირში ჩახაზვადი კონფიგურაციები. მიღებულია წრეწირში ჩახაზვადი სფერული ოთხკუთხედის ფართობის ფორმულა მისი გვერდების ტერმინებში. ნაჩვენებია აგრეთვე, რომ სფერულ სახსრულ ჯაჭვთან ასოცირებული ასახვა მდგრადია და მისი განსაკუთრებულობათა სიმრავლე შედგება რამდენიმე წრეწირისგან.

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