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Mathematics

On Poisson Type Integral Representations in a Unit Ball

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ABSTRACT. We consider Hardy classes of analytic functions in the unit ball of the space $\mathbb{C}^n$. Representations of functions belonging to a Hardy class by Poisson type integrals are established. Besides, it is shown that if $f \in L^p(\sigma)$ and the function satisfies certain conditions, then the operator $T[f] \in H^p(B)$ and the map $f \to T[f]^*$ is a linear bounded projector from $L^p(\sigma)$ onto $H^p(S)$, where $T[f]^*$ is the *K*-limit of $T[f]$ on the sphere *S*.

***Key words**: Hardy class, K-limit, Cauchy kernel, pluriharmonic function.*

We denote by $\mathbb{C}$ the set of complex numbers, and by $\mathbb{C}^n$ the set of all $z = (z_1, \cdots, z_n)$, where $z_j \in \mathbb{C}$, $z_j = x_j + iy_j$, $x_j, y_j \in R = (-\infty, +\infty)$, $j = \overline{1,n}$. The Euclidean norm of the vector $z = (z_1, \cdots, z_n)$ is denoted by $|z|$.

We denote by *B* and *S* respectively the unit ball of the space $\mathbb{C}^n$ and the sphere, i.e.

$$B = \left\{ z \in \mathbb{C}^n : |z| < 1 \right\},$$

$$S = \left\{ z \in \mathbb{C}^n : |z| = 1 \right\}.$$

The point $z = (z_1, \cdots, z_n)$ will also be written in the form $z = r \cdot t$, where $r = |z|$ and $t = \dfrac{z}{|z|}$, *r* and *t* are called the spherical coordinates of the point *z*.

We denote by σ the usual normed Lebesgue measure ($\sigma(S)=1$) on the sphere *S*.

Let $p>0$ be some number. We denote by $L^p(B)$ the space of summable functions of order p in *B*, and by $H(B)$ the set of all analytic functions in the ball *B*.

A function $f : B \to \mathbb{C}$ is called pluriharmonic if $f \in \mathbb{C}^2(B)$ and

$$D_k \overline{D}_j f = 0, \quad j = \overline{1,n}, \quad k = \overline{1,n},$$

where

$$D_k = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right), \quad \overline{D}_j = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_k}\right).$$

The set of pluriharmonic functions is denoted by $RP(B)$.

If $f : B \to \mathbb{C}$ and $0 < r < 1$, then we denote by f_r the function defined in a ball $|z| < \frac{1}{r}$ by the equality $f_r(z) = f(rz)$.

We will consider the following functions:

1) $C(z,t) = (1 - \langle z,t\rangle)^{-n}$, $z \in B, t \in S$;

2) $P(z,t) = \dfrac{(1-|z|^2)^n}{|1-\langle z,t\rangle|^{2n}}$, $z \in B,\ t \in S$;

3) $K(z,t) = 2\operatorname{Re} C(z,t) - 1$, $z \in B,\ t \in S$,

where $\langle z,t\rangle = \sum_{j=1}^{n} z_j t_j$, $z = (z_1, \cdots, z_n)$, $t = (t_1, \cdots, t_n)$.

Let us assume that $f \in L^p(B)$, $p \geq 1$ and consider the operators:

$$C[f](z) = \int_S C(z,t) f(t)\, d\sigma(t),$$

$$P[f](z) = \int_S P(z,t) f(t)\, d\sigma(t),$$

$$T[f](z) = \int_S K(z,t) f(z)\, d\sigma(t).$$

It is clear that $C[f] \in H(B)$, $P[f] \in RP(B)$ and $T[f] \in RP(B)$.

Take some number $\alpha > 1$ and a point $t \in S$. Consider the set

$$D_\alpha(t) = \left\{ z \in \mathbb{C}^n : |1 - \langle z,t\rangle| < \frac{\alpha}{2}(1 - |z|^2) \right\}.$$

It is obvious that $D_\alpha(t) \subset B$.

We say that the function $F : B \to \mathbb{C}$ has a number λ as its *K*-limit at a point $t \in S$, symbolically $(K - \lim F)(t) = \lambda$ if the equality

$$\lim_{z \to t} F(z) = \lambda, \text{ where } z \in D_\alpha(t),$$

is fulfilled for every $\alpha > 1$.

The *k*-limit of the function *F* at the point *t* is denoted by $F^*(t)$.

We denote by *A*(*B*) the set of functions analytic in *B* and continuous on the closure $\overline{B}$. It is obvious that *A*(*B*) is an algebra with respect to the Chebyshev norm.

Assume that *A*(*S*) is the set of all those functions $f \in C(S)$ which represent the restriction of functions of the algebra *A*(*B*) of the sphere *S*. It is clear that *A*(*S*) is the closed subalgebra of *C*(*S*) with respect to the Chebyshev (uniform) norm.

Assume that $0 < p < \infty$. Denote by $H^p(S)$ the closure L^p of the set *A*(*S*), i.e. $f \in H^p(S)$ means that there exists a sequence $f_n \in A(S)$ such that

$$\lim_{n\to\infty} \|f_n - f\|_p = 0 .$$

We say that a function $f \in H(B)$ belongs to the Hardy space $H^p(B)$ if

$$\sup_{0\le r<1} \int_S |f_r|^p d\sigma < +\infty , \quad 0 < p < \infty . \tag{1}$$

Denote

$$\|f\|_p = \left[\sup_{0\le r<1} \int_S |f_r|^p \, d\sigma \right]^{\frac{1}{p}} .$$

It is obvious that this number is the norm when $p \ge 1$ and the value $\|f - g\|_p^p$ defines the metric when $0 < p < 1$. The following theorems are valid (see. [1], pp. 94, 95).

Theorem 1. *If* $f \in H^p(B)$, *then f has a finite k-limit* f^* *almost everywhere on the sphere S.*

Theorem 2. *Assume that* $0 < p < \infty$, $f \in H^p(B)$ *and* $f^*(t) = (k - \lim f)(t)$ *almost everywhere on S. Then*

$$\lim_{r\to 1} \int_S \left| f_r - f^* \right| d\sigma = 0 . \tag{2}$$

Theorem 3. (*a*) *If* $f \in H^p(B)$, *then* $f^* \in H^p(S)$ *and* $\left\|f^*\right\|_p = \|f\|_p$; (*b*) *if* $p \ge 1$ *and* $g \in H^p(S)$, *then* $P[g] = C[g] \in H^p(B)$ *and* $g = C[g]^*$ *almost everywhere on S, where*

$$P[g](z) = \int_S P(z,t) g(t) d\sigma , \quad (z \in B,\ t \in S),$$

is the Poisson invariant integral for the unit sphere, and

$$C[g](z) = \int_S C(z,t) g(t) d\sigma , \quad (z \in B,\ t \in S)$$

is the Cauchy-Sege integral operator for the unit ball.

From Theorem 3 it follows that if $f \in H^p(B)$ and $p \ge 1$, then

$$P\left[f^*\right] = C\left[f^*\right] \in H^p(B).$$

Let us prove the theorems.

Theorem 4. *If* $f \in H^1(B)$ *then the following equality is valid:*

$$\int_S f^*(t) \left(\overline{\langle z,t \rangle}\right)^k d\sigma = 0 , \quad \forall z \in B,\ \forall k \in N .$$

Proof. We use the equality

$$\int_S f^*(t) \left(\overline{\langle z,t \rangle}\right)^k d\sigma = \int_S f_r(t) \left(\overline{\langle z,t \rangle}\right)^k d\sigma + \int_S \left[f^*(t) - f_r(t) \right] \left(\overline{\langle z,t \rangle}\right)^k d\sigma . \tag{3}$$

Suppose $f(z)=\sum_{m=0}^{\infty} b_m(z)$ is the homogeneous decomposition of the function *f* in the ball *B*. Since $b_m(z)$ and $\left(\langle z,t\rangle\right)^k$ are homogeneous orthogonal polynomials (see [1], p. 23), we have

$$\int_S f_r\left(\overline{\langle z,t\rangle}\right)^k d\sigma=\int_S\left[\sum_{m=0}^{\infty} b_m(rt)\left(\overline{\langle z,t\rangle}\right)^k d\sigma\right]=\sum_{m=0}^{\infty} r^m\int_S b_m(t)\left(\overline{\langle z,t\rangle}\right)^k d\sigma=0,\ \ \forall z\in B,\ \forall k\in N,\ m\neq n. \quad (4)$$

Now let us estimate the second summand in the right-hand part of equality (3):

$$\left|\int_S\left[f^*(t)-f_r\right]\left(\overline{\langle z,t\rangle}\right)^k d\sigma\right|\le\int_S\left|f_r-f^*\right|\left|\overline{\langle z,t\rangle}\right|^k d\sigma\le\int_S\left|f_r-f^*\right||z|^k|t|^k d\sigma\le\int_S\left|f_r-f^*\right|d\sigma. \quad (5)$$

From relations (3) – (5) we obtain

$$\left|\int_S f^*(t)\left(\overline{\langle z,t\rangle}\right)^k d\sigma\right|\le\int_S\left|f_r-f^*\right|d\sigma. \quad (6)$$

The left-hand side of inequality (6) does not depend on *r*, therefore if here we pass to the limit as $r\to 1$ and take into account Theorem 2, then we obtain

$$\int_S f^*(t)\left(\overline{\langle z,t\rangle}\right)^k d\sigma=0,\ \ \forall z\in B,\ \forall k\in N.$$

Since $H^p(B)\subset H^1(B)$ when $p\ge 1$, Theorem 4 is true for $f\in H^p(B)$.

Theorem 5. *If* $f\in H^p(B)$, $p\ge 1$, *then* $C\left[\overline{f^*}\right](z)=\overline{f(0)}$.

Proof. According to Theorem 3,

$$f(z)=C\left[f^*\right](z)=\int_S C(z,t)f^*(t)d\sigma,\ \ \forall z\in B,$$

which implies

$$f(0)=C\left[f^*\right](0)=\int_S f^*(t)d\sigma, \quad (7)$$

$$C\left[\overline{f^*}\right](z)=\int_S \overline{f^*}(t)C(z,t)d\sigma=\overline{\int_S f^*(t)\overline{C(z,t)}d\sigma}=$$

$$=\overline{\int_S f^*(t)\left[\sum_{k=0}^{\infty}\frac{\Gamma(k+n)}{k!\Gamma(n)}\left(\overline{\langle z,t\rangle}\right)^k\right]d\sigma}=\overline{\sum_{k=0}^{\infty}\frac{\Gamma(k+n)}{k!\Gamma(n)}\int_S f^*(t)\left(\overline{\langle z,t\rangle}\right)^k d\sigma}=\overline{\int_S f^*(t)d\sigma}=\overline{f(0)}.$$

Here we have used Theorem 4 and the equality $C(z,t)=\sum_{k=0}^{\infty}\frac{\Gamma(k+n)}{k!\Gamma(n)}\left(\langle z,t\rangle\right)^n$, where Γ is Euler's function.

Theorem 6. *If* $f\in H^p(B)$ $(1\le p<\infty)$, *then*

$$f(z)=T\left[f^*\right](z)=\int_S K(z,t)f^*(t)d\sigma,\ \ \forall z\in B.$$

Proof. By Theorem 3 $f(z)=C\left[f^*\right](z)$. Therefore we have

$$T\left[f^*\right](z)=\int_S\left[2\operatorname{Re}C(z,t)-1\right]f^*(t)\,d\sigma=\int_S\left[C(z,t)+\overline{C(z,t)}-1\right]f^*(t)\,d\sigma=$$

$$=\int_S C(z,t)f^*(t)\,d\sigma+\int_S f^*(t)\overline{C(z,t)}d\sigma-\int_S f^*(t)\,d\sigma\,. \tag{8}$$

By Theorem 5

$$\int_S f^*(t)\overline{C(z,t)}d\sigma=f(0). \tag{9}$$

Taking into account the formula $f(z)=C\left[f^*\right](z)$, from equalities (8) and (9) we obtain

$$T\left[f^*\right](z)=C\left[f^*\right](z)+f(0)-f(0)=C\left[f^*\right](z)=f(z).$$

Theorem 7. *Assume that* $f\in L^1(\sigma)$. *A function* $T[f]$ *is analytic in B if and only if the equalities*

$$\int_S f(t)\left(\overline{\langle z,t\rangle}\right)^k d\sigma=0,\quad \forall z\in B,\ \forall k\in N, \tag{10}$$

are fulfilled.

Proof. If equalities (10) are fulfilled, then, taking into account the equality,

$$\overline{C(z,t)}=\sum_{k=0}^{\infty}\frac{\Gamma(k+n)}{k!\Gamma(n)}\left(\overline{\langle z,t\rangle}\right)^k$$

we have

$$T[f](z)=\int_S\left[2\operatorname{Re}C(z,t)-1\right]f(t)d\sigma=\int_S\left[C(z,t)+\overline{C(z,t)}\right]f(t)\,d\sigma-$$

$$-\int_S f(t)d\sigma=\int_S C(z,t)f(t)d\sigma+\int_S\overline{C(z,t)}f(t)\,d\sigma-\int_S f(t)d\sigma=$$

$$=C[f](z)+\sum_{k=0}^{\infty}\frac{\Gamma(k+n)}{k!\Gamma(n)}\int_S f(t)\left(\overline{\langle z,t\rangle}\right)^k d\sigma-\int_S f(t)d\sigma=$$

$$=C[f](z)+\int_S f(t)d\sigma-\int_S f(t)d\sigma=C[f](z)=\int_S\frac{f(t)\,d\sigma}{\left(1-\langle z,t\rangle\right)^k}\,.$$

The latter is an analytic function in *B*. We have thereby proved the sufficiency of the condition.

Let us prove the necessity. Assume that $T[f](z)$ is an analytic function in *B*, then by virtue of the above transformations we obtain

$$T[f](z)=C[f](z)+\int_S f(t)C(z,t)\,d\sigma-\int_S f(t)d\sigma=C[f](z)+$$

$$+\sum_{k=0}^{\infty}\frac{\Gamma(k+n)}{k!\Gamma(n)}\int_S\left(\overline{\langle z,t\rangle}\right)^k f(t)d\sigma+\int_S f(t)d\sigma-\int_S f(t)\,d\sigma=C[f](z)+\sum_{k=1}^{\infty}\frac{\Gamma(k+n)}{k!\Gamma(n)}\int_S\left(\overline{\langle z,t\rangle}\right)^k f(t)d\sigma$$

or, which is the same,

$$T[f](z)-C[f](z)=\sum_{k=1}^{\infty}\frac{\Gamma(k+n)}{k!\Gamma(n)}\int_S\left(\overline{\langle z,t\rangle}\right)^k f(t)\,d\sigma .$$

The left-hand side of this equality is an analytic function in B, whereas the right-hand side is not, in general, analytic and therefore this equality will be fulfilled only when the function

$$\varphi(z)=\sum_{k=1}^{\infty}\frac{\Gamma(k+n)}{k!\Gamma(n)}\int_S\left(\overline{\langle z,t\rangle}\right)^k f(t)\,d\sigma$$

is constant in B, i.e. $\varphi(z)=const,\ \forall z\in B$. In particular, if z=0, then $\varphi(0)=0$. Thus

$$\sum_{k=1}^{\infty}\frac{\Gamma(k+n)}{k!\Gamma(n)}\int_S\left(\overline{\langle z,t\rangle}\right)^k f(t)\,d\sigma\equiv 0,\quad \forall z\in B .$$

However, this may take place only when

$$\int_S\left(\overline{\langle z,t\rangle}\right)^k f(t)\,d\sigma=0,\quad \forall k\in N .$$

The theorem is proved. From the above reasoning it follows that if a function $f\in L^1(\sigma)$ satisfies conditions (10), then

$$T[f](z)=C[f](z).$$

By the Korányi-Vagi theorem (see [1], p. 107) stating that if $1<p<\infty$ and $f\in L^p(\sigma)$, then $C[f]\in H^p(B)$ and the mapping $f\to C[f]^*$ is a linear bounded projection of the space $L^p(\sigma)$ onto the space $H^p(S)$), we conclude that if $1<p<\infty$ and a function $f\in L^p(\sigma)$ satisfies conditions (10), then $T[f]\in H^p(B)$ and the map $f\to T[f]^*$ is a linear bounded projection of $L^p(\sigma)$ onto the space $H^p(B)$, where $T[f]^*$ is the K-limit of $T[f]$, which exists under these conditions almost everywhere on S. Indeed, by virtue of the above reasoning we see that

$$T[f](z)=C[f](z)+\int_S\overline{C(z,t)}f(t)\,d\sigma-\int_S f(t)\,d\sigma=C[f](z)+\int_S\overline{\overline{f}C(z,t)}\,d\sigma-\int_S f(t)\,d\sigma=$$

$$=C[f](z)+\overline{C[\overline{f}](z)}-\int_S f(t)\,d\sigma . \tag{11}$$

But the functions $C[f]$ and $\overline{C[\overline{f}]}$ have the finite K-limit almost everywhere on S (see [1], Theorem 6.2.3, p. 216), therefore equality (11) implies that the function $T[f]$ has the finite K-limit $T[f]^*$ almost everywhere on the sphere S.

მათემატიკა

პუასონის ტიპის ინტეგრალური წარმოდგენის შესახებ ერთეულოვან ბირთვში

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განხილულია $\mathbb{C}^n$ სივრცის ერთეულოვან B ბირთვში ანალიზურ ფუნქციათა ჰარდის კლასები. დადგენილია ამ კლასის ფუნქციების წარმოდგენები პუასონის ტიპის ინტეგრალებით. ამასთანავე ნაჩვენებია, რომ თუ $f \in L^p(\sigma)$ და ფუნქცია აკმაყოფილებს გარკვეულ პირობებს, მაშინ ოპერატორი $T[f] \in H^p(B)$ და ასახვა $f \to T[f]^*$ არის წრფივი შემოსაზღვრული პროექტორი $L^p(\sigma)$-დან $H^p(S)$-ზე, სადაც $T[f]^*$ არის $T[f]$-ის *K*-ზღვარი *S*-ზე.

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