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Mathematics

On the Properties of Holomorphic Functions in some Bergman Weighted Spaces

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ABSTRACT. The paper discusses weighted Bergman spaces of holomorphic functions in the unit circle introduced by Matevosyan. Integral representations of functions of this class and their majorants are given. The case where a fractional integral belongs to the Hardy classes is studied. *© 2011 Bull. Georg. Natl. Acad. Sci.*

Key words: *Bergman space, Hardy class, analytical function, fractional integral.*

Let D be the unit circle on a complex plane and $\omega(r)$ be a monotone non-negative function in the space $L^1(0,1)$. Denote by $B^p(\omega)$ $(0<p<+\infty)$ the space of holomorphic functions $f:D\to C$ for which

$$\|f\|^p_{B^p(\omega)}=\int_0^1\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^p\omega(r)rdrd\theta<+\infty. \quad (1)$$

These spaces are the weighted variants of the well-known Bergman spaces and were introduced by P.A. Matevosyan in [1].

One of the basic results from [1] is the following

Theorem [1]. *Let* $\omega:(0,1)\to R_+$ *satisfy the condition*

$$\sup_{0<r<1}\left|\frac{\omega'(r)(1-r)}{\omega(r)}\right|=q_\omega<+\infty, \quad (2)$$

where it is assumed that $0<q_\omega<1$ *holds when* $\omega(r)$ *is an increasing function. If* $f(z)=\sum_{n=0}^{\infty}a_nz^n\in B^p(\omega)$ $(0<p\le 1)$, *then*

$$|a_n|=o\left(\frac{(n+1)^{\frac{2}{p}-1}}{\left[\omega\left(1-\frac{1}{n+1}\right)\right]^{\frac{1}{p}}}\right)\quad as\quad n\to+\infty. \quad (3)$$

Conversely, if $\varphi(n)$ *is an arbitrary non-negative function monotonically tending to zero as* $n \to +\infty$, *then there exists a function* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *that satisfies the conditions*

$$a)\ f \in B^p(\omega) \quad (0 < p \le 1), \qquad b)\ \overline{\lim_{n\to\infty}} \frac{|a_n| \left[\omega\left(1 - \frac{1}{n+1}\right)\right]^{\frac{1}{p}}}{(n+1)^{\frac{2}{p}-1} \varphi(n)} = +\infty .$$

In what follows, an entire function $\psi : C \to C$ is called a first order and minimal type function if for every $\varepsilon > 0$ there exists a positive number $A(\varepsilon)$ such that the inequality

$$|\psi(z)| < e^{\varepsilon |z|}$$

is valid for all $|z| > A(\varepsilon)$.

As is known, for each power series

$$f(z) = \sum_{n=0}^{\infty} a_n(f) z^n \tag{4}$$

with the unit circle of convergence there exists a majorant series

$$F(z) = \sum_{n=0}^{\infty} \psi(n) z^n, \quad |a_n(f)| < \psi(n), \quad n = \overline{0,\infty}, \tag{5}$$

having angular boundary values almost everywhere on the circumference and $\psi : C \to C$ is a first order and minimal type entire function ([2], p. 49).

There arises the question to which class the majorant function (5) belongs if the initial function (4) belongs to the space $B^p(\omega)$.

In this context, let us recall the theorem of Viehert-Lo ([2], p. 50): *The necessary and sufficient condition for the series (4) to be an entire function with respect to* $(1-z)^{-1}$ *is the existence of a first order and minimal type function* ψ *such that the equalities* $a_n(f) = \psi(n)$, $n = \overline{0,\infty}$, *are fulfilled.*

We want to answer the question whether for every function $B^p(\omega)$ there exists a majorant function of the form (5) which belongs to the Hardy class $H^p(D)$ $(0 < p < 1)$.

The answer to this question is positive and, moreover, the following theorems are valid.

Theorem 1. *If* $f \in B^p(\omega)$, $0 < p \le 1$, *where* $\omega(r)$ *satisfies the condition* (2) *and* $\omega(1-0) = \lim_{r\to 1^-} \omega(r) = c > 0$, *then for f there exists a majorant function*

$$F(z) = \sum_{n=0}^{\infty} \psi(n) z^n$$

which belongs to $H^\delta(D)$ *for every* $0 < \delta < \frac{1}{1 + [2p^{-1}]}$, *where* $\psi : C \to C$ *is an entire function of first order and minimal type.*

Here by [·] we denote the integer part.

Proof. Let $f(z)=\sum_{n=0}^{\infty}a_n(f)z^n\in B^p(\omega)$. Then by virtue of (3) we have

$$|a_n(f)|=o\left(\frac{(n+1)^{\frac{2}{p}-1}}{\left[\omega\left(1-\frac{1}{n+1}\right)\right]^{\frac{1}{p}}}\right).$$

From this it follows that

$$\lim_{n\to\infty}\left[\omega\left(1-\frac{1}{n+1}\right)\right]^{\frac{1}{p}}|a_n(f)|(n+1)^{-[2p^{-1}]}=0.$$

Therefore there exists a natural number $n_0\in N$ such that for $n>n_0$ we have

$$\left[\omega\left(1-\frac{1}{n+1}\right)\right]^{\frac{1}{p}}|a_n(f)|<(n+1)^{[2p^{-1}]}.$$

Denoting

$$M=\max_{0\le k\le n_0}\left\{\left[\omega\left(1-\frac{1}{k+1}\right)\right]^{\frac{1}{p}}|a_k(f)|\right\},$$

we have

$$\left[\omega\left(1-\frac{1}{n+1}\right)\right]^{\frac{1}{p}}|a_n(f)|<(n+1)^{[2p^{-1}]}+M+1,\quad n=\overline{0,\infty}. \tag{6}$$

Let

$$\beta_n=\left[\omega\left(1-\frac{1}{n+1}\right)\right]^{-\frac{1}{p}},$$

then by virtue of (6) we obtain

$$|a_n(f)|<\beta_n(n+1)^{[2p^{-1}]}+\beta_n(M+1),$$

where $\forall n\in N$, $\beta_n<\beta$. Such β exists by virtue of $\omega(1-0)=c>0$. It is obvious that the majorant function will be

$$F(z)=\sum_{n=0}^{\infty}\left[(n+1)^{[2p^{-1}]}+M+1\right]\beta z^n,\quad \psi(z)=\beta\left[(z+1)^{[2p^{-1}]}+M+1\right],$$

$$\psi(z)=\beta\left[(z+1)^{[2p^{-1}]}+M+1\right].$$

But this function belongs to the class $H^{\delta}(D)$ and the theorem is proved.

Theorem 2. *Let* $\omega(r)$ *satisfy the condition* (2) *and* $\omega(1-0)=c>0$. *If* $f\in B^p(\omega)$, $0<p\le 1$, *then it can be represented as*

$$f(z)=\frac{\left(1+\left[2p^{-1}\right]\right)!}{2\pi}\int_0^{2\pi}\frac{\varphi\left(e^{i\theta}\right)d\theta}{\left(1-ze^{-i\theta}\right)^{\left[2p^{-1}\right]+2}},\quad z=re^{it}, \tag{7}$$

where the angular boundary function $\varphi\left(e^{i\theta}\right)$ is square-integrable.

Indeed, multiplying both parts of the inequality (6) by $(n+1)$, we obtain

$$\left[\omega\left(1-\frac{1}{n+1}\right)\right]^{\frac{1}{p}}\left|a_n(f)\right|(n+1)<(n+1)^{\left[2p^{-1}\right]+1}+(M+1)(n+1). \tag{8}$$

The inequality (8) implies that there exists a natural number m such that for $n\geq m$ we have the inequality

$$\left[\omega\left(1-\frac{1}{n+1}\right)\right]^{\frac{1}{p}}\left|a_n(f)\right|(n+1)<(n+1)(n+2)\cdots\left(n+\left[2p^{-1}\right]+1\right)=\psi(n). \tag{9}$$

Let us construct a new function

$$\varphi(z)=\sum_{n=0}^{\infty}\frac{a_n(f)}{\psi(n)}z^n,$$

where $a_n(f)$ are the coefficients of the function f. From the relations

$$\|\varphi\|_{L^2[0,2\pi]}=\sum_{n=0}^{\infty}\left|\frac{a_n(f)}{\psi(n)}\right|^2=\sum_{n=0}^{m}\left|\frac{a_n(f)}{\psi(n)}\right|^2+\sum_{n=m+1}^{\infty}\left|\frac{a_n(f)}{\psi(n)}\right|^2<\sum_{n=0}^{m}\left|\frac{a_n(f)}{\psi(n)}\right|^2+\sum_{n=m+1}^{\infty}\frac{1}{\left[\omega\left(1-\frac{1}{n}\right)\right]^{\frac{2}{p}}(n+1)^2}<+\infty$$

it follows that $\varphi\in H^2(D)$ and has an angular boundary function $\varphi\left(e^{i\theta}\right)\in L^2[0,2\pi]$. Moreover, $\varphi(z)\cdot z^{\mu}\in H^2(D)$, $\mu=1+\left[2p^{-1}\right]$. On the other hand,

$$f(z)=\frac{d^{\mu}}{dz^{\mu}}\left[z^{\mu}\varphi(z)\right],\quad z\in D.$$

Therefore, writing $z^{\mu}\varphi(z)$ in the form of a Cauchy integral

$$z^{\mu}\varphi(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\mu t}\varphi\left(e^{it}\right)dt}{1-ze^{-it}}$$

and differentiating it μ times, we obtain

$$f(z)=\frac{\left(1+\left[2p^{-1}\right]\right)!}{2\pi}\int_0^{2\pi}\frac{\varphi\left(e^{it}\right)dt}{\left(1-ze^{-it}\right)^{\left[2p^{-1}\right]+2}}.\quad \text{Q.E.D.} \tag{10}$$

Following Riemann and Liouville, for the function $f(z)=\sum_{n=0}^{\infty}a_n(f)z^n$ let us define a fractional integral of order α by the equality

$$D_\alpha f(z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)}a_n z^n,\quad (\alpha>1).$$

For $\alpha=1+\left[2p^{-1}\right]$ we have

$$D_\alpha f(z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma\left(n+2+\left[2p^{-1}\right]\right)}a_n z^n. \tag{11}$$

By virtue of binomial expansion, from (10) we obtain

$$f(z)=\frac{\left(1+\left[2p^{-1}\right]\right)!}{2\pi}\int_0^{2\pi}\left[\sum_{n=0}^{\infty}\frac{\Gamma\left(n+2+\left[2p^{-1}\right]\right)}{\Gamma(n+1)}z^n e^{-\text{int}}\right]\varphi\left(e^{it}\right)dt=$$

$$=\frac{\left(1+\left[2p^{-1}\right]\right)!}{2\pi}\sum_{n=0}^{\infty}\left[\frac{\Gamma\left(n+2+\left[2p^{-1}\right]\right)}{\Gamma(n+1)}\int_0^{2\pi}\varphi\left(e^{it}\right)e^{-\text{int}}dt\right]z^n. \tag{12}$$

Since $\varphi\in H^2(D)$, the formulas (11), (12) imply

$$D_\alpha f(z)=\frac{\left(1+\left[2p^{-1}\right]\right)!}{2\pi}\sum_{n=0}^{\infty}\int_0^{2\pi}\varphi\left(e^{it}\right)e^{-\text{int}}z^n dt=\frac{\left(1+\left[2p^{-1}\right]\right)!}{2\pi}\int_0^{2\pi}\left[\sum_{n=0}^{\infty}e^{-\text{int}}z^n\right]\varphi\left(e^{it}\right)dt=$$

$$=\frac{\left(1+\left[2p^{-1}\right]\right)!}{2\pi}\int_0^{2\pi}\frac{\varphi\left(e^{it}\right)dt}{1-ze^{-it}}=\left(1+\left[2p^{-1}\right]\right)\varphi(z)\in H^2(D).$$

Consequently, the following statement is valid.

Theorem 3. *Let* $\omega\in L(0,1)$ *satisfy, in addition to* (2), *the condition* $\omega(1-0)=\lim_{r\to1^-}\omega(r)=c>0$. *If* $f\in B^p(\omega)$, $0<p\le1$, *then its fractional integral of order* $\alpha=1+\left[2p^{-1}\right]$ *belongs to the Hardy space* $H^2(D)$, *i.e.,* $\left(\lambda_n=\frac{\Gamma(n+1)}{\Gamma\left(n+2+\left[2p^{-1}\right]\right)}\right)_{n\ge1}$ *is the multiplier from* $B^p(\omega)$ *to* $H^2(D)$. *This means that if* $f(z)=\sum_{n=0}^{\infty}a_n(f)z^n\in B^p(\omega)$, $(0<p\le1)$, *then*

$$\sum_{n=0}^{\infty}\left[\frac{\Gamma(n+1)}{\Gamma\left(n+2+\left[2p^{-1}\right]\right)}\right]^2\left|a_n(f)\right|^2<+\infty.$$

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ზოგიერთ წონიან სივრცეში ჰოლომორფულ ფუნქციათა თვისებების შესახებ

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