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Mathematics

A Maximum Inequality for Rearrangements of Summands and its Applications to Orthogonal Series and Scheduling Theory

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ABSTRACT. We give a two-sided estimation for the average of the quantity $\Phi(\max_{k\le n}\| a_{\pi(1)} + \dots + a_{\pi(k)} \|)$ **with respect to all permutations** $\pi : \{1, 2, \dots n\} \to \{1, 2, \dots n\}$**, where** $a = (a_1, \dots, a_n)$ **is an arbitrary collection of elements of a normed space** ***X*****, and** $\Phi : R^+ \to R^+$ **is an arbitrary increasing convex function. In the case** $\Phi(t) = t^2$ **and** $X = R^1$ **or** $X = C^1$ **the upper part of the inequality coincides with the famous Garsia inequality having a series of applications in orthogonal series and other problems of analysis. We also give a constructive algorithm for finding an optimal permutation having applications in scheduling theory.** *© 2011 Bull. Georg. Natl. Acad. Sci.*

Key words: *rearrangement of summands, maximum inequality, Garsia inequality, Nikishin theorem, algorithm for an optimal permutation.*

1. Introduction

One of the main purposes of this paper is to prove a two-sided maximum inequality estimating the quantity $A(\Phi, a) = \frac{1}{n!}\sum_{\pi}\Phi(\max_{k\le n}\| a_{\pi(1)} + \dots + a_{\pi(k)} \|)$, where $a = (a_1, \dots, a_n)$ is a collection of elements of a normed space X, $\Phi : R^+ \to R^+$ is a convex increasing function, while π runs through all permutations of indices $\{1,\dots,n\}$.

The problem has a series of applications in analysis (investigation of the sum range of a conditionally convergent series [1], almost everywhere convergence of a functional series under an appropriate rearrangement ([2,3] and the bibliography therein), as well as in theoretical and even practical problems of scheduling theory [4,5].

The problem goes back to the works [6,7] of Garsia and has a long history (see [2] and the bibliography therein). The first two-sided estimations were found by Maurey and Pisier [8]. They have shown that when $\sum_1^n a_i = 0$ and $\Phi(x) = x^p$, $x \ge 0$, $1 \le p < \infty$ the quantity $A(\Phi, a)$ is equivalent to (i.e. upper and lower estimations between them hold true with absolute constants) $(E \, \|\sum_1^n a_i r_i\|^p)^{1/p}$, where (r_i), $i = 1,\dots,n$ are Rademacher random variables, and

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E is the operation of expectation. However their paper was written in a somewhat general setting of exchangeable random variables and for a long time remained unknown to specialists. The result was rediscovered in our paper [9].

In this paper we give a simple proof of the equivalence inequality for the general case of increasing convex Φ with no restriction on the sum $\sum_1^n a_i$. A drastic simplification of proof and improvement of coefficients (the latter is of importance for applications) is achieved due to elementary Lemma 1. We also consider a possibility of generalization of the equivalence (Theorem 1) to the case of an arbitrary increasing Φ (not necessarily convex). Finally we give an algorithm (which runs in a polynomial time) for finding a permutation π that guarantees the values $\max_{1\le k\le n} \| a_{\pi(1)} + ... + a_{\pi(k)} \| \le D+\varepsilon$ for some constant $D > 0$ and any fixed $\varepsilon > 0$, provided that there is an algorithm ensuring $\max_{1\le k\le n} \| a_{\sigma(1)}\vartheta_1 + ... + a_{\sigma(k)}\vartheta_k \| \le D$ for any permutation σ and some collection of signs $\vartheta = (\vartheta_1, ..., \vartheta_n)$.

2. Notations.

X denotes a normed space, real or complex with the norm $\|\cdot\|$. In the space X^n of all collections $a = (a_1,...,a_n)$, $a_i \in X$, $i = 1,...,n$ we use the norm

$$||| a |||_n = \max_{1\le k\le n} \| a_1 + ... + a_k \|.$$

Denote by Π_n the group of all permutations $\pi : \{1,...,n\} \to \{1,...,n\}$. Given $a = (a_1,...,a_n)$, a permutation $\pi \in \Pi_n$ and scalars $\vartheta = (\vartheta_1, ..., \vartheta_n)$, a_π will stand for $(a_{\pi(1)},...,a_{\pi(n)})$, and $a_\pi\vartheta$ for $(a_{\pi(1)}\vartheta_1,...,a_{\pi(n)}\vartheta_n)$. $r_1,...,r_n$ denote Rademacher random variables defined on $[0,1]$ with the Lebesgue measure λ on it, i.e. they are independent and take only two values -1 and +1 with equal probabilities ½. E denotes the expected value (integration with respect to λ).

3. The main theorem.

Theorem 1. *Let $a = (a_1,...,a_n)$ be a collection of elements of a normed space X, $s \equiv \sum_1^n a_i$, and let $\Phi : [0,\infty) \to [0,\infty)$ be an increasing convex function. Then the following two-sided inequality holds:*

$$E\Phi(\frac{1}{2}\| s + \sum_1^n a_i r_i \|) \le \frac{1}{n!}\sum_{\pi\in\Pi_n} \Phi(||| a_\pi |||) \le 2E\Phi(\| s + \sum_1^n a_i r_i \|). \quad (1)$$

The proof of Theorem 1 is based on the following Lemma 1.

Given a permutation $\pi \in \Pi_n$, $\pi = (k_1, ..., k_n)$ and $\vartheta \in \{-1,+1\}^n$ let us define the following permutation $\pi^* \in \Pi_n$: $\pi^* = (k_{p_1},...,k_{p_u}, k_{q_v}, k_{q_{v-1}},...,k_{q_1})$, where $k_{p_1}, ..., k_{p_u}$ correspond to $p_1 <, ..., < p_u$ for which $\vartheta_{p_j} = +1$, $j = 1,...,u$ and $k_{q_1},...,k_{q_v}$ correspond to $q_1 <, ..., < q_v$ for which $\vartheta_{q_j} = -1$, $j = 1,...,v$; $u + v = n$.

Lemma 1. *Let $(a_1,...,a_n)$ be a collection of elements of a normed space X, $s \equiv \sum_1^n a_i$. Then for each $\pi \in \Pi_n$ and $\vartheta \in \{-1,+1\}^n$ the following inequality holds*

$$||| -s,\ a_\pi |||_{n+1} + ||| -s,\ a_\pi\vartheta |||_{n+1} \ge 2\, ||| -s,\ a_{\pi^*} |||_{n+1}. \quad (2)$$

Proof. We have

$$||| -s,\ a_\pi |||_{n+1} + ||| -s,\ a_\pi\vartheta |||_{n+1} \ge 2 \max\{||| -s,\ a_\pi^+ |||_{n+1},\ ||| 0,\ a_\pi^- |||_{n+1}\}, \quad (3)$$

where a_π^+ is the ordered collection with zeros for the indices $l = k_{q_1},...,k_{q_v}$, and with a_l for $l = k_{p_1},...,k_{p_u}$, and a_π^- is the collection with zeros for $l = k_{p_1},...,k_{p_u}$ and with a_l for $l = k_{q_1},...,k_{q_v}$. Taking into account that $-s + \sum_1^n a_i = 0$, we get from (3) the desired inequality (2).

Lemma 1 has been stated in different versions and used in [2,9,10,11].

Proof of Theorem 1. To prove the right-hand side inequality we use Lemma 1 to get the following inequality

$$\sum_{\pi}\Phi(|||-s,\ a_{\pi}\,|||_{n+1})=\sum_{\pi}\Phi(|||-s,\ a_{\pi}\,|||_{n+1}\ +|||-s,\ a_{\pi}\vartheta|||_{n+1}-|||-s,\ a_{\pi}\vartheta|||_{n+1})\geq$$
$$\geq\sum_{\pi}[2\,\Phi(|||-s,\ a_{\pi^*}\,|||_{n+1})-\Phi(|||-s,\ a_{\pi}\theta|||_{n+1})]. \tag{4}$$

Let us remark that for a fixed ϑ π^* runs through the whole Π_n as π runs through the whole Π_n. Therefore, (4) gives

$$\sum_{\pi}\Phi(|||\ a_{\pi}\ |||_{n})=\sum_{\pi}\Phi(|||\ -s,\ a_{\pi^*}\ |||_{n+1})\leq\sum_{\pi}\Phi(|||\ -s,a_{\pi}\vartheta\,|||_{n+1}). \tag{5}$$

Summing up the left- and right-hand sides with respect to all ϑ-s we get

$$2^n\sum_{\pi}\Phi(|||\ a_{\pi}\ |||_{n})\ \leq\ \sum_{\vartheta}\sum_{\pi}\Phi(|||-s,\ \ a_{\pi}\vartheta\ |||_{n+1}). \tag{6}$$

And finally, making use of the Levy inequality we get the desired inequality:

$$\frac{1}{n!}\sum_{\pi}\Phi(|||\ a_{\pi}\ |||_{n})\leq\frac{1}{n!}\sum_{\pi}E\Phi(|||\ -s,\ a_{\pi}r\,|||_{n+1})\ \leq\ \frac{2}{n!}\sum_{\pi}E\Phi(||\ sr_0+\sum_1^n a_{\pi(i)}r_i\ ||)\ \leq\ 2E\Phi(||s+\sum_1^n a_i r_i\ ||).$$

Let us proceed to the left-hand side inequality. We have according to the triangle inequality

$$\frac{1}{2^n}\sum_{\vartheta}\Phi(\frac{1}{2}||\ -s+\sum_1^n a_i\vartheta_i\ ||)\ \leq\ \frac{1}{2^n n!}\sum_{\pi}\sum_{\vartheta}\Phi(\frac{1}{2}|||\ -s\ \ ,a_{\pi}\vartheta\,|||_{n+1}\ )\leq$$
$$\frac{1}{2^n n!}\sum_{\pi}\sum_{\vartheta}\Phi(\max\,(|||\ -s,\ \ a_{\pi}^{+}\,|||_{u+1},\,|||\,a_{\pi}^{-}\,|||_{v}))\ \ =\ \ \frac{1}{2\ n!}\sum_{\pi}\sum_{\vartheta}\Phi(|\,||-s,a_{\pi^*}\ ||_{n+1})\ \ =\ \ \frac{1}{n!}\sum_{\pi}\Phi(|||\,a\ \ \ |||_{n}).$$

4. Corollary: the Garsia inequality for orthogonal systems.

Consider a finite orthogonal system $\varphi_1,...,\varphi_n\in L_2(T,\Sigma,\nu)$. Taking $\Phi(x)=\ x^2,\ \ 0\leq\ x\ <\ \infty,$ we get by Theorem 1 that for any $t\ \in\ T$

$$\frac{1}{n!}\sum_{\pi}\max_{k\leq n}|\,\varphi_{\pi(1)}(t)+...+\ \ \varphi_{\pi(k)}(t)\,|^2\ \leq\ 2(|\sum\nolimits_1^n\varphi_i(t)\,|^2\ +\sum\nolimits_1^n|\,\varphi_i(t)\,|^2\ ).$$

Integrating both parts of this inequality with respect to the measure ν we come to the following famous Garsia inequality [6, 7]

$$\frac{1}{n!}\sum_{\pi}\int_T\max_{k\leq n}|\,\varphi_{\pi(1)}(t)+...+\varphi_{\pi(k)}(t)\,|^2\ \leq\ 4\sum\nolimits_1^n\int_T|\,\varphi_i(t)\,|^2d\nu(t). \tag{7}$$

Let us note that the considerable simplification of the proof as well as reduction of the constant (4 against 16) was achieved mainly by use of elementary Lemma 1.

We remark that the Garsia inequality (7) was used by Garsia as a main tool to prove his celebrated theorem on rearrangements of orthogonal series: *If* $(\varphi_n),\ \ n\in N$ *is an orthogonal system in* $L_2\ \ (T,\ \Sigma,\nu\ )$ *with* $\sum_1^\infty\int_T\left|\varphi_n\right|^2d\nu<\infty$*, then there exists a permutation* $\pi:\ N\rightarrow\ N$ *such that the series* $\sum_1^\infty\varphi_{\pi(n)}$ *converges* ν *almost everywhere (a.e.).*

The use of the same method allows to prove the following two-way generalization (to the case of vector-valued function and in general non-orthogonal) of the Garsia theorem .

Theorem 2. *Let a series* $\sum_1^\infty f_n$ *of Borel measurable functions,* $f_n:T\rightarrow\ X,\ \ n\in N$*, converges in measure* ν *(σ-finite) to a function* $s:\ T\rightarrow\ X$ *. If*

$$\text{the series } \sum_1^\infty f_n r_n \text{ converges } \nu \times\lambda\text{-a.e.} \tag{8}$$

then there is a permutation $\pi : N \to N$ *such that the series* $\sum_1^\infty f_{\pi(n)}$ *converges a.e. to s.*

When X=R^1 *or, X=C*^{1} *condition* (8) *is equivalent to*

$$\text{the series } \sum_1^\infty | f_n |^2 \text{ converges } \nu\text{-a.e.} \tag{9}$$

Theorem 2 for scalar functions *f* under condition (9) is known as the Nikishin theorem [12].

5. Generalizations of Theorem 1.

As we have noted, the left-hand side of (1) holds true for every increasing function Φ. The question of great interest is whether a sort of the right-hand side of (1) holds true for *increasing* Φ *without the assumption of convexity*. Apparently we can answer this question in the positive due to the following inequality we have found in [2] for the case of increasing function for the case of $\sum_1^n a_i = 0$. For any $t > 0$

$$P\{s : \| \sum_1^n a_i r_i(s) \| > 2t\} \le \frac{1}{n!} card\{\pi : ||| a_\pi ||| > t\} \le C_1 P\{s : \| \sum_1^n a_i r_i(s) \| > \frac{t}{C_2}\} ,$$

where C_1 and C_2 are absolute constants.

A related result of type (1) will be published separately.

6. Applications to the scheduling theory.

There is a series of problems of scheduling theory that reduce to the problem of minimizing $||| a_\pi |||_n$ given a collection $a = (a_1,...,a_n)$, and finding the minimizing permutation π [4]. In such a pure setting the problem can hardly be solved, especially on the algorithmic basis. Lemma 1 suggests the following algorithm which assumes that there is a *sign algorithm* of finding ϑ given π such that for some constant $D > 0$

$$||| -s,\ a_\pi \vartheta ||| \le D. \tag{10}$$

Let us rewrite the first inequality (2) of Lemma 1 in the form

$$||| a_\pi |||_n + ||| -s,\ a_{\pi_{inv}} \vartheta |||_{n+1} \ge 2\ ||| a_{\pi_1} |||_n, \tag{2'}$$

where $\pi = (k_1,...,k_n)$ is an arbitrary permutation, $\vartheta \in \{-1,+1\}^n$ is any collection of signs, $\pi_{inv} = (k_n,...,k_1)$, and $\pi_1 = (\pi_{inv})^*$.

At the first step we take an arbitrary π_0 and find ϑ_0 satisfying (10) for $\pi = \pi_0$. Then we use π_0 and ϑ_0 to find π_1 and θ_1, etc.

Theorem 3. *The following inequality holds true after the N-th iteration*

$$||| a_{\pi_N} |||_n \le (1 - \frac{1}{2^N}) D + \frac{1}{2^N} ||| a_{\pi_0} |||_n,$$

where π_0 *is an arbitrary starting permutation.*

Remark. For a general *d*-dimensional *X* a polynomial sign-algorithm with *D=2d* was constructed in [13], for $X = l_\infty^d$ with $D = \sqrt{2n \ \ln d}$ such an algorithm was described in [14]. Another algorithmic type result using a sign procedure was found earlier in [15]. A construction similar to that described in Theorem 3 was used in [5] where a slightly complicated form of Lemma 1 has been used.

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მათემატიკა

მაქსიმალური უტოლობა შესაკრებთა გადანაცვლებისათვის და მისი გამოყენება ორთოგონალურ მწკრივებსა და დაგეგმვის თეორიაში

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მოყვანილია ორმხრივი შეფასება $\Phi(\max_{k\le n}\| a_{\pi(1)} + \ldots + a_{\pi(k)} \|)$ სიდიდის საშუალოსათვის ყველა $\pi : \{1, 2, \ldots n\} \to \{1, 2, \ldots n\}$ გადანაცვლების მიმართ, სადაც $a = (a_1, \ldots, a_n)$ არის X ნორმირებული სივრცის ელემენტთა ნებისმიერი კრებული, ხოლო $\Phi : R^+ \to R^+$ არის ნებისმიერი ზრდადი ამოზნექილი ფუნქცია. როდესაც $\Phi(t) = t^2$ და $X = R^1$ ან $X = C^1$, ამ უტოლობის მარჯვენა ნაწილი ემთხვევა გარსიას ცნობილ უტოლობას, რომელსაც აქვს მთელი რიგი გამოყენებებისა ორთოგონალურ მწკრივებსა და ანალიზის სხვა საკითხებში. ჩვენ აგრეთვე მოგვყავს კონსტრუქციული ალგორითმი ოპტიმალური გადანაცვლების მოძებნის ამოცანისათვის, რომელსაც აქვს გამოყენება დაგეგმვის თეორიაში.

REFERENCES

1. *M.I. Kadets, V.M. Kadets* (1997), Series in Banach Spaces: Conditional and Unconditional Convergence. Birkhauser, Boston.
2. *S.A. Chobanyan* (1994), In: Hoffman-Jorgensen, J., Kuelbs, J., Markus, M.B. (eds.). Probability in Banach Spaces, 9. Birkhauser, Boston.
3. *G.J. Giorgobiani* (2000), Georgian Math. J., **7**, 1: 43-51.
4. *S.V. Sevost'yanov* (1994), Discrete Applied Mathematics, 55: 59-82.
5. *M. Makai* (2004), Appl. Math. Comp., 150: 785-801.
6. *A. Garsia* (1964), Ann. Math., 22: 286-309.
7. *A. Garsia* (1970), Topics in almost everywhere convergence. Markham, Chicago.
8. *B. Maurey, G. Pisier* (1975), Sem. Maurey-Schwartz 1974-1975, Annexe, Centre Math., Ecole Polytech., Paris.
9. *S.A. Chobanyan, G. Giorgobiani* (1989), Probability Theory on Vector Spaces IV, Proc. Conf., Lancut/Poland 1987, Lect. Notes Math., 1391: 33-46.
10. *G.D. Giorgobiani* (1988), Soobshch. Akad. Nauk Gruz.SSR, **130**, 3: 481-484.
11. *M-J. Chasco, S.A. Chobanyan* (1997), Mich. Math. J., **44**, 3: 607-617.
12. *E.M. Nikishin* (1967), Math. Notes, 14: 570-574.
13. *I. Barany, V.S. Grinberg* (1981), Linear Algebra Appl., 41: 49-62.
14. *J. Spencer* (1997), J. Comb. Theory, Ser. B., 23: 68-74.
15. *S.A. Chobanyan* (1987), Math. USSR-Sb., 56: 49-62.

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