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Mathematics

Coset Lattices of Lie Algebras and their Isomorphisms

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ABSTRACT. For the Lie algebra over the ring the lattice of cosets is constructed. Necessary and sufficient conditions for distributivity, modularity, semimodularity of coset lattices are found. The fundamental theorem of affine geometry for nilpotent of class 2 Lie algebras is proved. *© 2011 Bull. Georg. Natl. Acad. Sci.*

Key words: *Affine geometry, coset lattice, fundamental theorem.*

1. Introduction

In this paper, for the Lie algebra $\mathcal{L}$ over the ring K we construct the coset lattice $CL(\mathcal{L})$ and investigate the connections between the structure of A and $CL(\mathcal{L})$. These problems were posed by A. G. Kurosh and G. Birkhoff and studied by the theory of groups developed in M. Kurzio, N. V. Loyko, B. Bruno and in other works (see the monograph [1] and references in there).

In Section 2 the necessary and sufficient conditions are found for the distributivity, modularity, semimodularity of $CL(\mathcal{L})$, as well as for the decomposability of $CL(\mathcal{L})$ into a direct product and so on.

In common with the lattice of all subalgebras $L(\mathcal{L})$ having a source in geometric considerations (i.e., when $\mathcal{L}$ is a module over the ring K, $L(\mathcal{L})$ realizes the projective geometry $PG(\mathcal{L},K)$), the coset lattice $CL(\mathcal{L})$ also has its source in geometric considerations: when A is a torsion-free module over the domain K, the lattice $CL(\mathcal{L})$ realizes the affine geometry $AG(\mathcal{L},K)$ corresponding to the K-module $\mathcal{L}$.

In the mid 60ies a number of mathematicians concentrated their attention on investigating Lie algebras from the lattice standpoint [2-10]. A few years later the papers [11-23] appeared (see, also [24]). One of the principal objectives pursued by these investigations was to answer the question for which classes of Lie algebras the fundamental theorem of projective geometry is valid, that is to say, in which cases a lattice isomorphism is generated by a semilinear isomorphism. Examples show that for many classes this problem is answered negatively, especially when dealing with Lie algebras defined over fields. Hence in some cases it is advisable to consider a more concentrated lattice, say, the coset lattice $CL(\mathcal{L})$, than the lattice of subalgebras $L(\mathcal{L})$.

When studying isomorphisms of coset lattices $\varphi : CL(\mathcal{L})\to CL(\mathcal{L}_1)$, the one-to-one correspondence $\varphi : \mathcal{L}\to\mathcal{L}_1$ is constructed in the natural manner. Therefore among isomorphisms we should distinguish those for which $\varphi(0)=0$. Such isomorphisms will be called natural C-isomorphisms. We shall say that the fundamental theorem of affine geometry is valid for the algebra $\mathcal{L}$ over the ring K if any natural C-isomorphism is a semilinear isomorphism, with respect to the some ring isomorphism $h : K \to K_1$.

2. Some Restrictions on Coset Lattices

Let $\mathcal{L}$ be a Lie algebra over the ring K. We shall consider a set $CL(\mathcal{L})$ consisting of all cosets of A with respect to all subalgebras and of an empty set $\varnothing$. On $CL(\mathcal{L})$ we can introduce the following partial order: $X_1 \subset X_2 \Leftrightarrow X_1 \leq X_2$.

Proposition 1. *$CL(\mathcal{L})$ is a complete lattice; the operations "$\cup$" and "$\cap$" are defined as follows: $\bigcup_{\alpha\in J} U_\alpha$, $U_\alpha = a_\alpha + A_\alpha$, is the set-theoretic intersection; $\bigcup_{\alpha\in J} U_\alpha = \alpha_\beta + \langle A_\alpha, a_\alpha - a_\beta, \alpha \in J\rangle$, where β is some fixed index from J, A_α is a subalgebra of $\mathcal{L}$.*

Proof. Let $\{U_\alpha, \alpha \in J\} \subseteq CL(\mathcal{L})$. It is assumed that $\cap U_\alpha \neq \varnothing$. Then $\exists \alpha \in \mathcal{L}$ such that $a \in U_\alpha$, $\alpha \in J$. Therefore $U_\alpha = a + A_\alpha$ for any α. We shall show that $a + \left(\bigcap_{\alpha\in J} A_\alpha\right)$ is an exact lower bound. Let b+B be a class such that $b + B \leq a + A_\beta$ for any $\beta \in J$. Then we have $b - a \in A_\beta$ and $B \leq A_\beta$. Indeed, $b \in a + A_\beta \Rightarrow b = a + c$, $c \in A_\beta \Rightarrow b - a \in A_\beta$.

Let $d \in B$. We have $b + d \in a + A_\beta \Rightarrow b + d = a + c$, $c \in A_\beta \Rightarrow d = (a - b) + c$, $d \in A_\beta$. Clearly, $B \subseteq \bigcap_{\alpha\in J} A_\alpha$, $b - a \in \bigcap_{\alpha\in J} A_\alpha$. Therefore

$$b + B \subseteq b + \left(\bigcap_{\alpha\in J} A_\alpha\right) = a + (b - a) + \bigcap_{\alpha\in J} A_\alpha = a + \bigcap_{\alpha\in J} A_\alpha + \bigcap_{\alpha\in J} U_\alpha$$

and thus $a + \bigcap_{\alpha\in J} A_\alpha$ is an exact lower bound. If however this intersection is empty, then we shall regard $\varnothing$ as an exact lower bound. Let us prove the existence of an exact upper bound. Fix any index $\beta \in J$. Consider the subalgebra $\langle A_\alpha, a_\alpha - a_\beta, \alpha \in J\rangle$. It is obvious that $a_\gamma + A_\gamma \leq a_\beta + \langle A_\alpha, a_\alpha - a_\beta\rangle$ is an upper bound for any $\gamma \in J$. Let b+B be a class such that $\forall \gamma \in J$ we have $a_\gamma + A_\gamma \leq b + B$. Since $a_\gamma - b \in B$ and $A_\gamma \leq B$, we obtain

$$a_\alpha - a_\beta = (a_\alpha - b) - (a_\beta - b) \in B \Rightarrow \langle A_\alpha, a_\alpha - a_\beta, a \in J\rangle = \Omega \leq B \Rightarrow$$
$$\Rightarrow a_\beta + \Omega \leq a_\beta + B \Rightarrow a_\beta + \Omega \leq b + (a_\beta - b) + B \Rightarrow a_\beta + \Omega \leq b + B.$$

Hence $a_\beta + \Omega$ is an exact upper bound. It is obvious that it does not matter how the index β is chosen.

Proposition 2. *Let $\mathcal{L}$ be a Lie algebra over the ring K. The lattice $CL(\mathcal{L})$ is decomposable into the direct product if and only if $K \cong Z_2$ and $\mathcal{L}$=1.*

Proof. Let us consider the lattice isomorphism $\varphi : CL(\mathcal{L}) \to \bar{L} = L_1 + L_2$.

Since $\varnothing, A \in CL(\mathcal{L})$, in $\bar{L}, L_1, L_2$ the biggest and the smallest elements exist. Let these be $\bar{E}, \bar{O} \in \bar{L}$, $E_1, O_1 \in L_1$, $E_2, O_2 \in L_2$. We assume that

$$\varphi(b + B) = (E_1, O_2) = \bar{E}_1 \in \bar{L}, \quad \varphi(c + C) = (O_1, E_2) = \bar{E}_2 \in \bar{L}.$$

It is obvious that for any $\bar{x} = (x_1, x_2) \in \bar{L}$ we have $\bar{X} = (\bar{X} \cap \bar{E}_1) \cup (\bar{X} \cap \bar{E}_2)$. Therefore for any $a \in A$ we obtain $a = [a \cap (b + B)] \cup [a \cap (c + C)]$ from which it follows that $\mathcal{L} = (b + B) \cup (c + C)$. Let us show that $B = C$. There exists $a \in \mathcal{L}$ for which $a + B \leq c + C$. Indeed, let us assume the opposite. Then $(a + B) \cap (b + B) \neq \varnothing$, i.e.,

$a+b_1=b+b_2 \Rightarrow a\in b+B$ for any $a\in \mathcal{L}$. Therefore

$$b+B=A\Rightarrow B=A\Rightarrow \exists\ a, a+B\le c+C\Rightarrow B\le C.$$

Similarly, $C\le B$ and therefore $C=B$. Further we have $A=(b+B)\bigcup(c+B)$. This results in $0\in b+B$ or $0\in c+B$.

If $0\in b+B$, then $b\in B$. It is likewise clear that $c\notin B$, since otherwise $\mathcal{L}=B$ and $\mathcal{L}=B\bigcup C$. We have to show that B is maximal in $\mathcal{L}$. For this it is enough to show that B is a maximal subgroup of the abelian group $\mathcal{L}$. Assume that there exists a subgroup G such that $B\subset G\subset \mathcal{L}$. Then there is $g\in G$, $g\notin B$. On the other hand, $\mathcal{L}=B\bigcup(c+B)$ implies

$$g\in c+B,\quad g=c+u,\quad u\in B\Rightarrow c=g-u, g,\quad u\in G,\quad c\in G.$$

Since $B\subset G$ and $c\in G$, we obtain $B\bigcup C=G$, i.e., $\mathcal{L}\subseteq G$. Therefore $G=\mathcal{L}$ and B is a maximal subgroup and hence a maximal subalgebra in $\mathcal{L}$.

Let us now assume that either the lattice L_1 or L_2 contains more than two elements. Then there exists $Y_1\in L_1$ such that $E_1\supset Y_1\supset O$. In $\bar{L}$ consider the chain $(O_1,E_2)\subset(Y_1,E_2)\subset(E_1,E_2)$ to which for φ^{-1} there corresponds the chain

$$\varphi^{-1}[(O_1,E_2)]=b+B\subset\varphi^{-1}[(Y_1,E_2)]=f+F\subset\varphi^{-1}[(E_1,E_2)]=\mathcal{L}.$$

This means that we have strict imbeddings $B\subset F\subset \mathcal{L}$, which is impossible. Therefore L_1 and L_2 contain only two elements each; thus $CL(\mathcal{L})$ contains four elements, which proves that $K\cong Z_2$ and $\dim\mathcal{L}=1$.

Proposition is proved.

Coset lattices of Lie algebras usually have not nice lattice properties. For example, if A is a proper subalgebra of a Lie algebra $\mathcal{L}$, $O\subset A\subset\mathcal{L}$ and $x\in\mathcal{L}\setminus A$, then $\{\varnothing, x, x+A, A, \langle x, A\rangle\}$ is a nonmodular sublattice (pentagon) of $CL(\mathcal{L})$ (see Fig. 1).

Thus algebra has no proper subalgebras, i.e. is defined over a field and $\dim\mathcal{L}=1$. So, modular coset lattice of a Lie algebra $\mathcal{L}$ has the form (Fig. 2), where only the elements of $\mathcal{L}$ are the atoms in $CL(\mathcal{L})$.

It is clear that $CL(\mathcal{L})$ is distributive if and only if when the lattice in Fig. 2 contains only two atoms, i.e. $\dim\mathcal{L}=1$, $K=Z_2$.

Now let $CL(\mathcal{L})$ be lower semimodular and Lie algebra $\mathcal{L}$ is defined over the principal ideal domain K. In this case a maximal subalgebra $X\subset\mathcal{L}$ and an element $a\notin X$ exist. Therefore $\mathcal{L}=X\bigcup(a+X)$ and $\mathcal{L}$ covers X; hence, by the condition $a+X$ covers $X\bigcap(a+X)=\varnothing$. Thus $X=0$ and so we conclude that K is a field and $\dim\mathcal{L}=1$. Thus we have

Theorem 1. *Let $\mathcal{L}$ be a Lie algebra over the principal ideal domain K, then*

(i) $CL(\mathcal{L})$ *is distributive if and only if* $\dim\mathcal{L}=1$, $K=Z_2$;

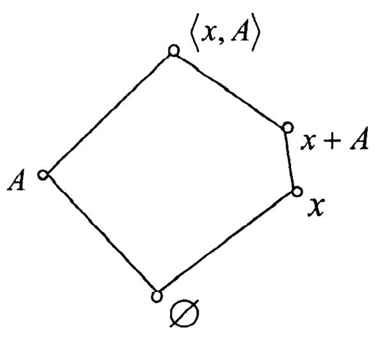

Fig. 1.

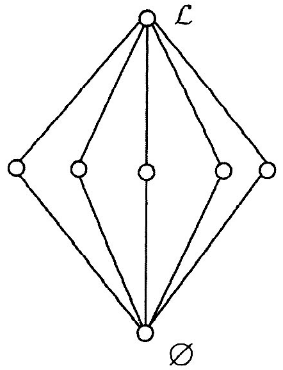

Fig. 2.

(ii) $CL(\mathcal{L})$ *is modular if and only if K is a field and* $\dim \mathcal{L} = 1$;

(iii) $CL(\mathcal{L})$ *is lower semimodular if and only if it is modular.*

3. Complements and Chain Conditions

Now we assume that $\mathcal{L}$ is a Lie algebra is defined over a field F.

Proposition 3. *If K is a field, then* $CL(\mathcal{L})$ *is the lattice with complements.*

Proof. Let $a+X \in CL(\mathcal{L})$ and Y be a maximal subalgebra in A that contains X and the element $b \notin Y$. Then $\mathcal{L} = \langle b \rangle \bigcup Y$; $Y \bigcap (b+Y) = \varnothing$. Since $X \subseteq Y$, we have

$$X \bigcap (b+Y) \subseteq Y \bigcap (b+Y) = \varnothing \Rightarrow (a+Y) \bigcap \left[(a+b)+Y\right] = \varnothing .$$

On the other hand,

$$(a+X) \bigcap \left[(a+b)+Y\right] = a + \langle X, Y, b = (a+b) - a \rangle = \mathcal{L},$$

i.e., $(a+X) \bigcap \left[(a+b)+Y\right]$ is the biggest element in $CL(\mathcal{L})$.

Proposition 4. *If K is a field, then* $CL(\mathcal{L})$ *is the lattice with relative complements if and only if* $L(\mathcal{L})$ *is such.*

Proof. The necessity follows from the fact that any closed interval of the lattice with relative complements is the lattice with relative complements itself and also from the fact that $L(\mathcal{L}) \cong [\varnothing, \mathcal{L}]$, i.e., $L(\mathcal{L})$ coincides with the interval $[\varnothing, \mathcal{L}]$ in the lattice $CL(\mathcal{L})$.

To prove the sufficiency we consider an arbitrary interval $[U,V] \subseteq L(\mathcal{L})$, where $U = a+X$, $V = b+Y$. If $U = \varnothing$, we have $[U,V] = [\varnothing, b+Y] \cong [\varnothing, Y] \cong L(\mathcal{L})$.

Since $L(\mathcal{L})$ is the lattice with complements, the interval $[U,V]$ will also be such. Let $U = \varnothing$. Then $[U,V] = [a+X, b+Y] \cong [X,Y]$.

Since $L(\mathcal{L})$ is the lattice with relative complements, we find that the intervals $[X,Y]$ and therefore $[U,V]$ are the lattices with complements.

Thus each interval in the lattice $CL(\mathcal{L})$ is with complements, i.e., $CL(\mathcal{L})$ is the lattice with relative complements.

Proposition 5. *If K is a field, then the condition of Jordan-Dedekind is fulfilled in the lattice* $CL(\mathcal{L})$ *if and only if it is fulfilled in* $L(\mathcal{L})$.

Proof. The necessity follows from the fact that $L(\mathcal{L})$ coincides with the interval $[\varnothing, \mathcal{L}] \subset CL(\mathcal{L})$.

To prove the sufficiency we assume that $U = a+X$, $V = b+Y$, $U \supset V$.

Then there are nonsaturated chains

$$U = A_0 \supset \cdots \supset A_{n-1} \supset A_n = V, \tag{*}$$

$$U = B_0 \supset B_1 \supset \cdots \supset B_{m-1} \supset B_m = V. \tag{**}$$

Furthermore, each element $\mu \in \mathcal{L}$ defines the automorphism $\tilde{\mu} \in Aut\left[CL(\mathcal{L})\right]$ as follows:

$$\tilde{\mu} : CL(\mathcal{L}) \to CL(\mathcal{L}), \quad \tilde{\mu} : x \to x + \mu, \quad \tilde{\mu}(U) = U + \mu .$$

The automorphism $\tilde{a}$ $(a \in \mathcal{L})$ (*) and (**) into the nonsaturated chains connecting X and Y. Therefore $m=n$. If however $V=\varnothing$, then it can be assumed that $A_{n-1}=a$, $B_{m-1}=b$. Then $U=a+X=b+Y$, and applying the automorphism $\tilde{\alpha}=(-\tilde{a})$, we obtain a nonsaturated chain of length n-1 from the chain (*), and a nonsaturated chain of length m-1 from the chain (**). Both chains connect X and (in our case) $\varnothing=Y$. Therefore $m-1=n-1\Rightarrow n=m$.

4. Affine Geometry and Natural C-Isomorphisms

Like the projective geometry $PG(X,K)$ corresponding to the torsion-free K-module X, where K is a principal ideal domain, is algebraically interpreted as the lattice $L(X)$ of all K-submodules [25]; the affine geometry $AG(X,K)$ corresponding to K-module X is the coset lattice $CL(X)$.

Let A and A_1 be linear algebras over the rings K and K_1, respectively. The bijection $f:A\rightarrow A_1$ will be called a semilinear quasi-isomorphism with respect to the isomorphism $h:K\rightarrow K_1$ if the equalities

$$f(\alpha x+\beta y)=h(\alpha)f(x)+h(\beta)f(y),\quad f(xy)=\lambda f(x)f(y),$$

$const=\lambda\in K_1$, $\lambda\neq 0$ are fulfilled for any $x_1,y_2\in A$, $\alpha,\beta\in K$.

The mapping f is called a semilinear isomorphism for $\lambda=1$ and a semilinear antiisomorphism for $\lambda=-1$.

Example 1. If σ is an automorphism of the ring A, then the mapping transferring the element $(\sigma(x_i))$ to each element (x_i) of the module A_s^n is a semilinear isomorphism $A_s^n\rightarrow A_s^n$ with respect to $\sigma\in Aut(A)$; we recall that A_s is the principal ring A considered as the left module over itself.

Example 2. Let now the ring A be noncommutative; for any a not belonging to the center of A the homothety defined by the mapping $x\rightarrow ax$ will not be, generally speaking, the linear mapping of the A-module E into itself. However, if a is invertible, then this homothety is a semilinear isomorphism with respect to the internal automorphism $b\rightarrow aba^{-1}$ of the ring A because $a(bx)=(aba^{-1})(ax)$.

We shall say that the fundamental theorem of projective geometry is valid for the algebra $\mathcal{L}$ over the ring K if a lattice isomorphism $\varphi:L(\mathcal{L})\rightarrow L(\mathcal{L}_1)$, where $\mathcal{L}_1$ is the algebra over the ring K_1, implies the existence of a semilinear isomorphism $g:L\rightarrow\mathcal{L}_1$ with respect to the isomorphism $h:K\rightarrow K_1$ such that the equality $\varphi(A)=g(A)$ holds for any subalgebra $A\in L(\mathcal{L})$.

Since in the lattice $CL(\mathcal{L})$ only elements $\mathcal{L}$ cover $\varnothing$, the isomorphism $f:CL(\mathcal{L})\rightarrow CL(\mathcal{L}_1)$ defines the bijection $\mathcal{L}\rightarrow\mathcal{L}_1$.

If $f:CL(\mathcal{L})\rightarrow CL(\mathcal{L}_1)$ is an isomorphism, then φ defined by the equality $\varphi(x)=f(x)-f(0)$ will be a natural C-isomorphism. We shall say that the fundamental theorem of affine geometry is valid for the algebra A if any natural C-isomorphism is a semilinear isomorphism.

Remark 1. If $a\in\mathcal{L}$ is a fixed element and $f(a)=a_1$, then the mapping $\varphi(x)=f(a+x)-a_1$ is a natural C-isomorphism. Indeed, φ will be a C-isomorphism defined by the element $-a_1$, i.e., it will be an automorphism $(-\tilde{a}_1)\in Aut[CL(\mathcal{L})]$; since $\varphi(0)=f(a)-a_1$, f will be a natural C-isomorphism.

Example 3. Not each natural C-isomorphism is a semilinear isomorphism. Any one dimensional space over Z_p admits $(p-1)!$ natural C-automorphisms while the group of internal automorphisms Z_p has the order $p-1$. Therefore for $p>3$ one-dimensional spaces over Z_p admit natural C-automorphisms different from ordinary ones.

Proposition 6. *Let $f:CL(\mathcal{L})\rightarrow CL(\mathcal{L}_1)$ be a natural C-isomorphism. Then the following statements are true:*

(1) *f induces a lattice isomorphism $f:L(\mathcal{L})\rightarrow L(\mathcal{L}_1)$;*

(2) $f(\langle M\rangle)=\langle f(M)\rangle$ *for any subset* $M\subseteq\mathcal{L}$;

(3) *if K and* K_1 *are principal ideal domains, then* $f(a+\langle b\rangle)=f(a)+\langle f(b)\rangle$, $a,b\in\mathcal{L}$;

(4) $f(\mu\ a)=\mu_1 f(a)$, $\mu_1\in K_1$, *is fulfilled for any* $a\in\mathcal{L}$ *and* $\mu\in K$;

(5) *if a and b are linearly independent elements and K and* K_1 *are principal ideal domains, then* $f(a)$ *and* $f(b)$ *are also such.*

Proof. (1) The lattice $L(\mathcal{L})$ coincides with the interval $[0,\mathcal{L}]\subset CL(\mathcal{L})$. In the case of a natural *C*-isomorphism this lattice is mapped onto the interval $[0,\mathcal{L}_1]\subset CL(\mathcal{L}_1)$, i.e., onto $L(\mathcal{L}_1)$. One should only keep in mind that the unions in the lattices $CL(\mathcal{L})$ and $L(\mathcal{L})$ coincide.

(2) $f(\langle M\rangle)=f(0\cup M)=f(0)\cup f(M)=0\cup f(M)=\langle f(M)\rangle$;

(3) $(a+\langle b\rangle)\cup 0=\langle a,b\rangle\Rightarrow f(a+\langle b\rangle)\cup f(0)=\langle f(a),f(b)\rangle$;

$$a\in a+\langle b\rangle\Rightarrow\varphi(a)\in\varphi(a+\langle b\rangle)\Rightarrow\varphi(a+\langle b\rangle)=\varphi(a)+M_1.$$

Therefore

$$(f(a)+M_1)\cup 0=\langle f(a),M_1\rangle\Rightarrow f(a+\langle b\rangle)\cap 0=\langle f(a),M_1\rangle\Rightarrow\langle f(a),f(b)\rangle=\langle f(a),M_1\rangle.$$

It is obvious that $M_1=\langle b_1\rangle$ is one-dimensional. Thus $f(a+\langle b\rangle)=f(a)+\langle b\rangle$ and

$$b_1=\alpha_1 f(a)+\beta_1(b),\ \alpha_1,\beta_1\in K_1,\text{ i.e., } f(a+\langle b\rangle)=f(a)+\langle\alpha_1 f(a)+\beta_1 f(b)\rangle.$$

Let $\alpha_1\neq 0$. We have

$$f(a)-(\alpha_1)^{-1}(\alpha_1 f(a)+\beta_1 f(b))\in f(a+\langle b\rangle)\Rightarrow-(-\alpha_1)^{-1}\beta_1 f(b)\in\Psi(a+\langle b\rangle)\Rightarrow$$
$$\Rightarrow\langle f(b)\rangle\cap f(a+\langle b\rangle)\neq\varnothing\Rightarrow\langle b\rangle\cap(a+\langle b\rangle)\neq\varnothing\ ;$$

(4) We have

$$\langle\mu a\rangle\subset\langle a\rangle\Rightarrow f(\langle\mu a\rangle)f(\langle a\rangle)\Rightarrow\langle f(\mu a)\rangle\subseteq\langle f(a)\rangle\Rightarrow f(\mu a)=\mu_1 f(a),\quad\mu\in K_1;$$

(5) If $f(a)$ and $f(b)$ are linearly dependent, then there exists $c_1\in A$ such that $\langle f(a),f(b)\rangle=\langle c_1\rangle$. Therefore

$$f^{-1}(\langle f(a),f(b)\rangle)=f^{-1}(\langle c_1\rangle)\Rightarrow\langle a,b\rangle=\langle c\rangle,\quad c_1=f(c).$$

The classical version of the fundamental theorem of affine geometry can be found in [19]. For the ring generalizations see [20, 27] and references in there. The Theorem 4 from [26] states the following:

Theorem 2 (Fundamental Theorem of Affine Geometry). *If* $\varphi: CL({}_RM)\to CL({}_{R_1}M_1)$, $\varphi(0)=0$ *is a lattice isomorphism, where* ${}_RM$ *and* ${}_{R_1}M_1$ *are free modules over the rings R and* R_1 *and* $\dim{}_RM\geq 2$, *then there exists an isomorphism* $\sigma: R\to R_1$ *such that the restriction* φ *on* ${}_RM$ *is a* σ*-semilinear isomorphism.*

Remark 2. The requirement that dim $A>1$ is essential. Indeed, for the one-dimensional space over the field *K* the lattice $CL(A)$ has the form in Fig. 2.

Therefore, any one-to-one correspondence $\varphi: A\to A_1$, where A_1 is the space over the field K_1 having the same cardinality as *K*, will be an *C*-isomorphism.

5. The Fundamental Theorem of Affine Geometry for Lie Algebras

Assume now that *K* is a commutative domain. Lie algebra *L* over *K* is called torsion-free if $\alpha x=0$ $(\alpha\in K,\ \ x\in L)$

implies $\alpha = 0$ or $x = 0$.

Lemma 1. *Let* $\dim L \geq 2$, $f : CL(\mathcal{L}) \to CL(\mathcal{L}_1)$ *be a natural C-isomorphism between torsion-free nilpotent Lie algebras over the rings K and* K_1. *Then:*

(a) $f(Z(\mathcal{L})) = Z(f(\mathcal{L}))$;

(b) *the nilpotency classes of* $\mathcal{L}$ *and* $\mathcal{L}_1$ *coincide;*

(c) *there exists an isomorphism* $h : K \to K_1$ *such that* $f(\mu b) = h(\mu) f(b)$, $\forall\ b \in \mathcal{L}$.

Proof. (a) Let $z \in Z(\mathcal{L})$, $a \in \mathcal{L}$, $\langle z \rangle \cap \langle a \rangle = 0$. Then $\dim\left[f(\langle a \rangle \cup \langle z \rangle)\right] = \dim\left[f(\langle a \rangle) \cup f(\langle z \rangle)\right] \leq 2$, i.e., $f(Z(\mathcal{L})) \leq Z(\mathcal{L}_1)$. For the inverse isomorphism f^{-1} we have

$$f^{-1}\left[Z(\mathcal{L}_1)\right] \leq Z(\mathcal{L}) \Rightarrow f\left[f^{-1}(Z(\mathcal{L}_1))\right] = Z(\mathcal{L}_1) \leq f(Z(\mathcal{L})) \Rightarrow f(Z(\mathcal{L})) = Z(\mathcal{L}_1).$$

(b) The center of a nilpotent algebra is isolated, i.e., we have the torsion-free algebra $L/Z(\mathcal{L})$. f induces a lattice isomorphism between $L/Z(\mathcal{L})$ and $L_1/Z(\mathcal{L}_1)$. The induction with respect to the nilpotence of the class enables us to conclude that the statement is true.

(c) Let $a \in \mathcal{L}$, $z \in Z(\mathcal{L})$, $\langle a \rangle = 0$. The subalgebras $A = \langle a \rangle \cup \langle z \rangle$ and $A_1 = \langle f(a) \rangle \cup \langle f(z) \rangle$ are abelian.

The natural C-isomorphism $f : CL(\mathcal{L}) \to CL(\mathcal{L}_1)$ is an h-semilinear isomorphism (Theorem 2), i.e., for any $\mu \in K$, $f(\mu z) = h(\mu) f(z)$, $f(\mu a) = h(\mu) f(a)$. If however $b = \alpha z$, then

$$f(\mu b) = f\left[\mu(\alpha z)\right] = f\left[(\mu\alpha) z\right] = h(\mu\alpha) f(z) = h(\mu) h(\alpha) f(z) = h(\mu) f(\alpha z).$$

Proposition 7. *Let L and* L_1 *be nilpotent Lie algebras over the fields F and* F_1, *and f be a natural C-isomorphism. Then* $f(a+b) = f(a) + f(b)$ *for any* $a, b \in L$.

Proof. First assume that $F \neq Z_2$. Consider the natural C-isomorphism $\varphi(x) = f(x-b) + f(b)$. We have

$$\varphi(a+b) = f(a+b-b) + f(b) = f(a) + f(b) \Rightarrow \varphi\left[2(a+b)\right] = 2\varphi(a+b) = 2f(a) + 2f(b) \Rightarrow$$
$$\Rightarrow f\left[2(a+b) - b\right] + f(b) = 2f(a) + 2f(b) \Rightarrow f(2a+b) = f(2a) + f(b) \Rightarrow$$
$$\Rightarrow f\left[2\left(\frac{1}{2}a\right) + b\right] = f\left(2\left(\frac{1}{2}a\right)\right) + f(b) = f(a) + f(b).$$

Let now $F = Z_2$. Then $a = -a$, $f(a) = -f(a)$. Using the Proposition 6, we have

$$\begin{array}{ccc} f\left[a \cup (a+b)\right] = f(a + \langle b \rangle) = & f\left[\begin{Bmatrix} a, \\ a+b \end{Bmatrix}\right] \\ \| & \\ f(a) \cup f(a+b) & \| \\ \| & \\ f(a) + \langle f(b) \rangle = \begin{Bmatrix} f(a), \\ f(a)+f(b) \end{Bmatrix} = & \begin{Bmatrix} f(a), \\ f(a+b) \end{Bmatrix} \end{array}$$

Consequently, $f(a+b) = f(a) + f(b)$.

Lemma 2. *Under the conditions of Lemma* 1 *if for any* $a, b \in L$, $[a, b] = c$ *we have* $\left[f(a), f(b)\right] = \mu f(c)$, *then*

$\mu = \text{const} \in K_1$.

Proof. Let $[a, b_1] = c_1$, $[a, b_2] = c_2$. Then

$$[f(a), f(b_1)] = \mu_1 f(c_1), \quad [f(a), f(b_2)] = \mu_2 f(c_2),$$
$$[a, b_1 + b_2] = c_1 + c_2 \Rightarrow [f(a), f(b_1 + b_2)] = \mu f(c_1 + c_2) = \mu f(c_1) + \mu f(c_2),$$
$$[f(a), f(b_1) + f(b_2)] = [f(a), f(b_1)] + [f(a), f(b_2)] = \mu_1 f(c_1) + \mu_2 f(c_2) \Rightarrow$$
$$\Rightarrow (\mu - \mu_1) f(c_1) = (\mu_2 - \mu) f(c_2)$$

We introduce the notation $\alpha_1 = \mu - \mu_1$, $\alpha = \mu_2 - \mu$. If $\alpha_1 = 0$ and $\alpha_2 = 0$, then $\mu_1 = \mu_2 = m$. If $\alpha_1 = 0$ and $\alpha_2 \neq 0$, then $f(c_2) = 0$. Therefore we may take any element instead of μ_2, i.e. $\mu_2 = \mu_1$. The situation $\alpha_1 \neq 0$, $\alpha_2 = 0$ is treated similarly. Consider the case $\alpha_1 \neq 0$, $\alpha_2 \neq 0$. There exists an isomorphism $f(\mu x) = h(\mu) f(x)$ such that $\bar{\alpha}_1, \bar{\alpha}_2 \in K$. Let $h(\bar{\alpha}_1) = \alpha_1$ be such that $h(\bar{\alpha}_1) = \alpha_1$, $h(\bar{\alpha}_2) = \alpha_2$. We have

$$\bar{\alpha}_1 c_1 = \bar{\alpha}_2 c_2, \quad \bar{\alpha}_2 [a, b_2] \Rightarrow [a, \bar{\alpha}_1 b_1 - \bar{\alpha}_2 b_2] = 0 \Rightarrow$$
$$\Rightarrow [f(a), \alpha_1 f(b_1) - \alpha_2 f(b_2)] = 0 \Rightarrow$$
$$\Rightarrow \alpha_1 [f(a), f(b_1)] = \alpha_2 [f(a), f(b_2)] \Rightarrow$$
$$\Rightarrow \alpha_1 \mu_1 f(c_1) = \alpha_2 \mu_2 f(c_2) \Rightarrow \mu_1 f(\alpha_1 c_1) = \mu_2 f(\alpha_2 c_2), \quad \mu_1 = \mu_2.$$

Theorem 3. *Let $f: CL(\mathcal{L}) \to CL(\mathcal{L}_1)$ be a natural C-isomorphism between the nilpotent of class 2 Lie algebras over the fields K and K_1, then f is a semilinear quasi-isomorphism with respect to the isomorphism $h: K \to K_1$.*

Proof. First we shall prove the theorem for the class of nilpotence 2. On account of the above arguments we can show that there exists $\mu \in K$ such that $f([a,b]) = \mu [f(a), f(b)]$. Everything is clear if *a*, *b* commute. Otherwise we shall have a natural *C*-isomorphism of 2-nilpotent algebras

$$f: CL(\langle a \rangle \cup \langle b \rangle) \to CL(\langle f(a) \rangle \cup \langle f(b) \rangle).$$

It is clear that

$$Z(\langle a \rangle \cup \langle b \rangle) = \langle [a,b] \rangle, \quad Z(\langle f(a) \rangle \cup \langle f(b) \rangle) = [f(a), f(b)].$$

Therefore

$$f[Z(\langle a \rangle \cup \langle b \rangle)] = z(\langle f(a) \rangle \cup \langle f(b) \rangle) \Rightarrow f([a,b]) = \varepsilon [f(a), f(b)].$$

Thus we have shown that the theorem holds for 2-nilpotent Lie algebras.

The given Example 4 shows that the theorem is false for the class of nilpotency ≥3.

Example 4. Let Lie algebra $\mathcal{L}$ over the field *F* is generated by the elements *a*, *b* and has the defining relations

$$0 \neq [a,b] = z, \quad [a,z] = [b,z] = 0.$$

It is clear that dim $\mathcal{L}$=4, $\mathcal{L}$ is nilpotent of class 3 and $Z(\mathcal{L}) = \langle [a,b] \rangle$. For arbitrary element

$$l = \alpha a + \beta b + \gamma [a,b] + \mu \quad z \in \mathcal{L}, \quad \alpha, \beta, \gamma, \mu \in F$$

consider the map $f: \mathcal{L} \to \mathcal{L}$

$$f(l) = \alpha a + \beta b + \gamma([a,b] + z) + \mu z.$$

We have

$$\alpha f\left(\left[a,b\right]\right)+\beta f\left(\left[a,b\right]\right)=\alpha\left(\left[a,b\right]+z\right)+\beta\left(\left[a,b\right]+z\right)=\left(\alpha+\beta\right)\left(\left[a,b\right]+z\right)=\left(\alpha+\beta\right)f\left(\left[a,b\right]\right)=$$
$$=f\left[\left(\alpha+\beta\right)\left[a,b\right]\right]=f\left(\alpha\left[a,b\right]+\beta\left[a,b\right]\right).$$

So for any

$$l_1=\alpha_1 a+\beta_1 b+\gamma_1\left[a,b\right]+\mu_1 z,\quad \alpha_1,\beta_1,\gamma_1,\mu_1\in F,$$
$$l_2=\alpha_2 a+\beta_2 b+\gamma_2\left[a,b\right]+\mu_2 z,\quad \alpha_2,\beta_2,\gamma_2,\mu_2\in F,$$

we have

$$f\left(l_1+l_2\right)=f\left(l_1\right)+f\left(l_2\right),\quad f\left(\alpha l\right)=\alpha f\left(l\right),\quad \alpha\in F.$$

Therefore

$$f\left(l_1\cup l_2\right)=f\left(l_1+\langle l_1-l_2\rangle\right)=f\left(l_1\right)+f\left(\langle l_1-l_2\rangle\right)=f\left(l_1\right)+\langle f\left(l_1-l_2\right)\rangle=$$
$$=f\left(l_1\right)+\langle f\left(l_1\right)-f\left(l_2\right)\rangle=f\left(l_1\right)\cup f\left(l_2\right).$$

Consequently, f is a natural A-automorphism of the lattice $CL(\mathcal{L})$, which is not a semilinear automorphism of $\mathcal{L}$.

Remark 3. From the Theorem 3 and Example 4 we can conclude that the Theorem 3 from [20] needs corrections, i.e. it is valid for nilpotency of class ≤3.

So the fundamental theorem of affine geometry for nilpotent of class ≥3 Lie algebras over the field is false.

The similar problems for Hall's W-power groups are considered in [28, 29].

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მათემატიკა

ლის ალგებრების მოსაზღვრე კლასების მესერები და მათი იზომორფიზმები

ა. ლაშხი

საქართველოს ტექნიკური უნივერსიტეტი

(წარმოდგენილია აკადემიკოს ხ. ინასარიძის მიერ)

კომუტატურ რგოლებზე განსაზღვრული ლის ალგებრებისათვის იგება მოსაზღვრე კლასების მესერები. ამ მესერებისათვის დისტრიბუციულობის, მოდულარულობის, ნახევრადმოდულარულობის აუცილებელი და საკმარისი პირობებია ნაპოვნი. 2-კლასის ნილპოტენტური ლის ალგებრებისათვის დამტკიცებულია აფინური გეომეტრიის ძირითადი თეორემა. აგებულია მაგალითი, რომელიც აჩვენებს, რომ თეორემა — *n*-ნილპოტენტურია *n*≥3 ლის ალგებრებისათვის — არ არის სამართლიანი.

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